Function spaces: Let L be the space of functions on $[0, \infty]$ equipped with norm $\|\cdot\|_L$. For any function u on $[0, \infty]$, let u_τ denote the truncation

$$u_\tau(t) = \begin{cases} u(t), & t \leq \tau \\ 0, & t > \tau \end{cases}$$

And define the extended space L_e as the set of all functions such that the truncation $u_\tau \in L$ for all $\tau \geq 0$. It is clear that $L \subset L_e$.

Input-output stability: In Khalil, a nonlinear input-output system, denoted by H, is called L-stable with finite gain if there are positive constants $\gamma, \beta > 0$ such that

$$\|(Hu)_\tau\|_L \leq \gamma\|u_\tau\|_L + \beta, \quad \forall u \in L_e, \quad \forall \tau \geq 0 \tag{0.1}$$

We want to compare this definition to the definition

$$\|Hu\|_L \leq \gamma\|u\|_L + \beta, \quad \forall u \in L \tag{0.2}$$

where the truncation is removed.

Assumptions: We make the following assumptions:

1. for $u \in L_e$, $\|u_\tau\|_L$ is a non-decreasing function of τ
2. for $u \in L$, $\|u_\tau\|_L \to \|u\|_L$ as $\tau \to \infty$
3. H is causal, i.e. $(Hu)_\tau = (Hu_\tau)_\tau$ for all $u \in L_e$ and $\tau \geq 0$.

The assumption seems to be true for all L_p norms.

Equivalency: We show that the two definitions are equivalent under these assumptions. First, let's assume (0.1) is true and we conclude (0.2). Let $u \in L$. Then, $u \in L_e$. Therefore, by (0.1)

$$\|(Hu)_\tau\|_L \leq \gamma\|u_\tau\|_L + \beta \leq \gamma\|u\|_L + \beta$$

where we used $\|u_\tau\|_L < \|u\|_L < \infty$ in the second inequality. The LHS of inequality is non-decreasing and bounded from above. Therefore, it has a limit which is equal to $\|Hu\|_L$. Therefore, as $\tau \to \infty$

$$\|Hu\|_L \leq \gamma\|u\|_L + \beta$$

which concludes (0.2).

To show the other direction, assume (0.2) is true. Take $u \in L_e$. Therefore, $u_\tau \in L$ for all $\tau \geq 0$. By application of (0.2) on u_τ

$$\|Hu_\tau\|_L \leq \gamma\|u_\tau\|_L + \beta$$

Note that $\|Hu_\tau\|_L \geq \|(Hu_\tau)_\tau\|_L = \|(Hu)_\tau\|_L$ where the first inequality follows from assumption 1 and the identity follows from causality. Therefore,

$$\|(Hu)_\tau\|_L = \|(Hu_\tau)_\tau\|_L \leq \|Hu\tau\|_L \leq \gamma\|u_\tau\|_L + \beta$$

which concludes (0.1).

Conclusion: The two definitions seems to be equivalent under the assumptions. And the assumptions seems to be valid for L_p norms that we are interested in. Unless there is a gap in the proof and there is some technicality that was missed. It will be great to see examples that satisfy the stability definition (0.2) but not (0.1).