Error Analysis of the Linear Feedback Particle Filter

American Control Conference, Milwaukee, June, 2018

Amirhossein Taghvaei
Joint work with P. G. Mehta

Coordinated Science Laboratory
University of Illinois at Urbana-Champaign

June 28, 2018

ILLINOIS

Outline

- Filtering problem in linear Gaussian setting
- Feedback Particle Filter (FPF)
- Stochastic and deterministic linear FPF
- Relation to the Ensemble Kalman filter
- Error analysis results
- Conclusion

Model:

State process: $\mathrm{d}X_t = AX_t\,\mathrm{d}t + \sigma_B\,\mathrm{d}B_t, \quad X_0 \sim \mathcal{N}(m_0, \Sigma_0)$

Observation process: $\mathrm{d}Z_t = HX_t\,\mathrm{d}t + \mathrm{d}W_t,$

Filtering objective: Find prob. of X_t given $\mathcal{Z}_t := \{Z_s; s \in [0,t]\}$

Kalman filter: $P_{X_t|\mathcal{Z}_t}$ is Gaussian $N(m_t, \Sigma_t)$

Mean: $$\mathrm{d}m_t = \underbrace{Am_t\,\mathrm{d}t}_{\text{propagation}} + \underbrace{\mathsf{K}_t(\,\mathrm{d}Z_t - Hm_t\,\mathrm{d}t)}_{\text{correction}}$$

Variance: $$\frac{\mathrm{d}\Sigma_t}{\mathrm{d}t} = \mathsf{Ric}(\Sigma_t) \text{ (Ricatti equation)}$$

Kalman gain: $$\mathsf{K}_t := \Sigma_t H^\top$$

J. Xiong, An introduction to stochastic filtering theory, 2008.
R. E Kalman and R. S Bucy. New results in linear filtering and prediction theory, 1961

Model:

State process: $\mathrm{d}X_t = AX_t\,\mathrm{d}t + \sigma_B\,\mathrm{d}B_t, \quad X_0 \sim \mathcal{N}(m_0, \Sigma_0)$

Observation process: $\mathrm{d}Z_t = HX_t\,\mathrm{d}t + \mathrm{d}W_t,$

Filtering objective: Find prob. of X_t given $\mathcal{Z}_t := \{Z_s; s \in [0,t]\}$

Kalman filter: $P_{X_t|\mathcal{Z}_t}$ is Gaussian $N(m_t, \Sigma_t)$

Mean: $$\mathrm{d}m_t = \underbrace{Am_t\,\mathrm{d}t}_{\text{propagation}} + \underbrace{\mathsf{K}_t(\,\mathrm{d}Z_t - Hm_t\,\mathrm{d}t)}_{\text{correction}}$$

Variance: $$\frac{\mathrm{d}\Sigma_t}{\mathrm{d}t} = \mathsf{Ric}(\Sigma_t) \text{ (Ricatti equation)}$$

Kalman gain: $$\mathsf{K}_t := \Sigma_t H^\top$$

J. Xiong, An introduction to stochastic filtering theory, 2008.
R. E Kalman and R. S Bucy. New results in linear filtering and prediction theory, 1961

Model:

State process: $\mathrm{d}X_t = AX_t\,\mathrm{d}t + \sigma_B\,\mathrm{d}B_t, \quad X_0 \sim \mathcal{N}(m_0, \Sigma_0)$

Observation process: $\mathrm{d}Z_t = HX_t\,\mathrm{d}t + \mathrm{d}W_t,$

Filtering objective: Find prob. of X_t given $\mathcal{Z}_t := \{Z_s; s \in [0,t]\}$

Kalman filter: $P_{X_t|\mathcal{Z}_t}$ is Gaussian $N(m_t, \Sigma_t)$

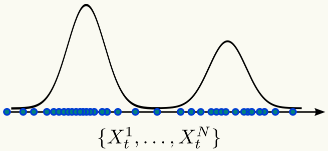
Mean: $$\mathrm{d}m_t = \underbrace{Am_t\,\mathrm{d}t}_{\text{propagation}} + \underbrace{\mathsf{K}_t(\mathrm{d}Z_t - Hm_t\,\mathrm{d}t)}_{\text{correction}}$$

Variance: $$\frac{\mathrm{d}\Sigma_t}{\mathrm{d}t} = \mathsf{Ric}(\Sigma_t) \text{ (Ricatti equation)}$$

Kalman gain: $$\mathsf{K}_t := \Sigma_t H^\top$$

J. Xiong, An introduction to stochastic filtering theory, 2008.
R. E Kalman and R. S Bucy. New results in linear filtering and prediction theory, 1961

A controlled interacting particle system to approximate the posterior dist.



Numerical experiments:

- Stano, et. al. (2013)
- Tilton, et. al. (2013)
- Berntorp, et. al. (2015)
- Surace, et. al. (2017)

(Yang, et. al. 2013)

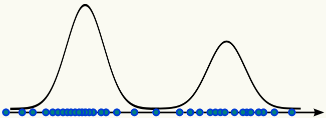
This work: Error analysis of the FPF algorithm for linear Gaussian setting

T. Yang, P. G. Mehta, and S. P. Meyn. feedback particle filter, *TAC*, 2013

T. Yang, R. S. Laugesen, P. G. Mehta, and S. P. Meyn. Multivariable feedback particle filter, *Automatica*, 2016

Feedback particle filter (FPF)
Overview

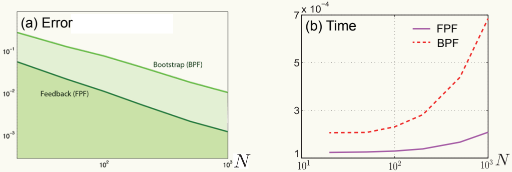
A controlled interacting particle system to approximate the posterior dist.



$\{X_t^1, \ldots, X_t^N\}$

Numerical experiments:

- Stano, et. al. (2013)
- Tilton, et. al. (2013)
- Berntorp, et. al. (2015)
- Surace, et. al. (2017)

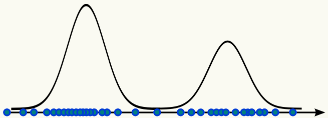


(Yang, et. al. 2013)

This work: Error analysis of the FPF algorithm for linear Gaussian setting

T. Yang, P. G. Mehta, and S. P. Meyn. feedback particle filter, *TAC*, 2013
T. Yang, R. S. Laugesen, P. G. Mehta, and S. P. Meyn. Multivariable feedback particle filter, *Automatica*, 2016

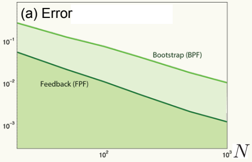
A controlled interacting particle system to approximate the posterior dist.

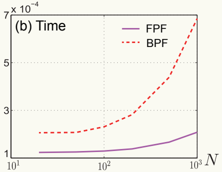


$\{X_t^1, \ldots, X_t^N\}$

Numerical experiments:

- Stano, et. al. (2013)
- Tilton, et. al. (2013)
- Berntorp, et. al. (2015)
- Surace, et. al. (2017)





(Yang, et. al. 2013)

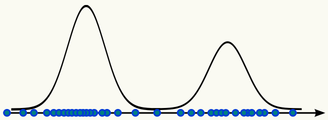
This work: Error analysis of the FPF algorithm for linear Gaussian setting

T. Yang, P. G. Mehta, and S. P. Meyn. feedback particle filter, *TAC*, 2013

T. Yang, R. S. Laugesen, P. G. Mehta, and S. P. Meyn. Multivariable feedback particle filter, *Automatica*, 2016

- **Particles:** $\{X_t^1, \ldots, X_t^N\}$
- **Mean-field process:** $\bar{X}_t$

$$\underbrace{\mathsf{E}[f(X_t)|\mathcal{Z}_t] = \mathsf{E}[f(\bar{X}_t)|\mathcal{Z}_t]}_{\text{exactness}} \approx \frac{1}{N}\sum_{i=1}^{N} f(X_t^i)$$

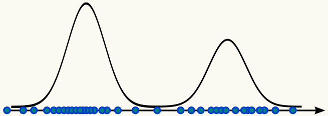


$\{X_t^1, \ldots, X_t^N\}$

T. Yang, P. G. Mehta, and S. P. Meyn. feedback particle filter, *TAC*, 2013
T. Yang, R. S. Laugesen, P. G. Mehta, and S. P. Meyn. Multivariable feedback particle filter, *Automatica*, 2016

- **Particles:** $\{X_t^1, \ldots, X_t^N\}$
- **Mean-field process:** $\bar{X}_t$

$$\underbrace{\mathsf{E}[f(X_t)|\mathcal{Z}_t] = \mathsf{E}[f(\bar{X}_t)|\mathcal{Z}_t]}_{\text{exactness}} \approx \frac{1}{N}\sum_{i=1}^{N} f(X_t^i)$$

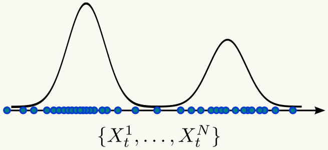


$\{X_t^1, \ldots, X_t^N\}$

T. Yang, P. G. Mehta, and S. P. Meyn. feedback particle filter, *TAC*, 2013
T. Yang, R. S. Laugesen, P. G. Mehta, and S. P. Meyn. Multivariable feedback particle filter, *Automatica*, 2016

- **Particles:** $\{X_t^1, \ldots, X_t^N\}$
- **Mean-field process:** $\bar{X}_t$

$$\underbrace{\mathsf{E}[f(X_t)|\mathcal{Z}_t] = \mathsf{E}[f(\bar{X}_t)|\mathcal{Z}_t]}_{\text{exactness}} \approx \frac{1}{N}\sum_{i=1}^{N} f(X_t^i)$$



T. Yang, P. G. Mehta, and S. P. Meyn. feedback particle filter, *TAC*, 2013
T. Yang, R. S. Laugesen, P. G. Mehta, and S. P. Meyn. Multivariable feedback particle filter, *Automatica*, 2016

Stochastic linear FPF: [Bergemann, et. al. 2012] [Yang, et. al. 2013]

$$\mathrm{d}\bar{X}_t = \underbrace{A\bar{X}_t\,\mathrm{d}t + \sigma_B\,\mathrm{d}\bar{B}_t}_{\text{propagation}} + \underbrace{\bar{\mathsf{K}}_t(\,\mathrm{d}Z_t - \frac{H\bar{X}_t + H\bar{m}_t}{2}\,\mathrm{d}t)}_{\text{correction (feedback control)}}, \quad \bar{X}_0 \sim p_0$$

Deterministic linear FPF: [Taghvaei, et. al. 2016]

$$\mathrm{d}\bar{X}_t = A\bar{X}_t\,\mathrm{d}t + \frac{1}{2}\sigma_B\sigma_B^\top\bar{\Sigma}_t^{-1}(\bar{X}_t - \bar{m}_t)\,\mathrm{d}t + \bar{\mathsf{K}}_t(\,\mathrm{d}Z_t - \frac{H\bar{X}_t + H\bar{m}_t}{2}\,\mathrm{d}t)$$

where the mean-field terms are

$$\bar{m}_t := \mathsf{E}[\bar{X}_t|\mathcal{Z}_t] \text{ (mean of } \bar{X}_t\text{)}$$

$$\bar{\Sigma}_t := \text{(covariance of } \bar{X}_t\text{)}$$

$$\bar{\mathsf{K}}_t := \bar{\Sigma}_t H^\top$$

G. Evensen. Sequential data assimilation with a nonlinear quasi-geostrophic model using monte carlo methods to forecast error statistics. 1994.
K. Bergemann and S. Reich. An ensemble Kalman-Bucy filter for continuous data assimilation, 2012
A. Taghvaei, P. G. Mehta, An optimal transport formulation for the linear feedback particle filter, (ACC) 2016

Stochastic and deterministic linear FPF
Mean-field limit

Stochastic linear FPF: [Bergemann, et. al. 2012] [Yang, et. al. 2013]

$$\mathrm{d}\bar{X}_t = A\bar{X}_t\,\mathrm{d}t + \sigma_B\,\mathrm{d}\bar{B}_t + \bar{\mathsf{K}}_t(\,\mathrm{d}Z_t - \frac{H\bar{X}_t + H\bar{m}_t}{2}\,\mathrm{d}t), \quad \bar{X}_0 \sim p_0$$

Deterministic linear FPF: [Taghvaei, et. al. 2016]

$$\mathrm{d}\bar{X}_t = A\bar{X}_t\,\mathrm{d}t + \frac{1}{2}\sigma_B\sigma_B^\top\bar{\Sigma}_t^{-1}(\bar{X}_t - \bar{m}_t)\,\mathrm{d}t + \bar{\mathsf{K}}_t(\,\mathrm{d}Z_t - \frac{H\bar{X}_t + H\bar{m}_t}{2}\,\mathrm{d}t)$$

where the mean-field terms are

$$\bar{m}_t := \mathsf{E}[\bar{X}_t|\mathcal{Z}_t] \text{ (mean of } \bar{X}_t\text{)}$$

$$\bar{\Sigma}_t := \text{(covariance of } \bar{X}_t\text{)}$$

$$\bar{\mathsf{K}}_t := \bar{\Sigma}_t H^\top$$

G. Evensen. Sequential data assimilation with a nonlinear quasi-geostrophic model using monte carlo methods to forecast error statistics. 1994.

K. Bergemann and S. Reich. An ensemble Kalman-Bucy filter for continuous data assimilation, 2012

A. Taghvaei, P. G. Mehta, An optimal transport formulation for the linear feedback particle filter, (ACC) 2016

Stochastic linear FPF:

$$\mathrm{d}X_t^i = AX_t^i\,\mathrm{d}t + \sigma_B\,\mathrm{d}B_t^i + \mathsf{K}_t^{(N)}\left(\mathrm{d}Z_t - \frac{HX_t^i + Hm_t^{(N)}}{2}\,\mathrm{d}t\right), \quad X_0^i \overset{\text{i.i.d}}{\sim} p_0$$

for $i = 1, \ldots, N$

Deterministic linear FPF:

$$\mathrm{d}X_t^i = AX_t^i\,\mathrm{d}t + \frac{1}{2}\sigma_B\sigma_B^\top {\Sigma_t^{(N)}}^{-1}(X_t^i - m_t^{(N)})\,\mathrm{d}t + \mathsf{K}_t^{(N)}\left(Z_t - \frac{HX_t + Hm_t^{(N)}}{2}\,\mathrm{d}t\right)$$

where the mean-field terms are empirically approximated

$$m_t^{(N)} := \frac{1}{N}\sum_{i=1}^{N} X_t^i \text{ (empirical mean)},$$

$$\Sigma_t^{(N)} := \frac{1}{N-1}\sum_{i=1}^{N}(X_t^i - m_t^{(N)})(X_t^i - m_t^{(N)})^\top \text{ (empirical covariance)}$$

Objectives:

- Convergence $m_t^{(N)} \to m_t, \quad \Sigma_t^{(N)} \to \Sigma_t$
- Convergence of the empirical distribution

Stochastic linear FPF:

$$\mathrm{d}X_t^i = AX_t^i\,\mathrm{d}t + \sigma_B\,\mathrm{d}B_t^i + \mathsf{K}_t^{(N)}(\mathrm{d}Z_t - \frac{HX_t^i + Hm_t^{(N)}}{2}\,\mathrm{d}t), \quad X_0^i \overset{\text{i.i.d}}{\sim} p_0$$

for $i = 1, \ldots, N$

Deterministic linear FPF:

$$\mathrm{d}X_t^i = AX_t^i\,\mathrm{d}t + \frac{1}{2}\sigma_B\sigma_B^\top {\Sigma_t^{(N)}}^{-1}(X_t^i - m_t^{(N)})\,\mathrm{d}t + \mathsf{K}_t^{(N)}(Z_t - \frac{HX_t + Hm_t^{(N)}}{2}\,\mathrm{d}t)$$

where the mean-field terms are empirically approximated

$$m_t^{(N)} := \frac{1}{N}\sum_{i=1}^N X_t^i \text{ (empirical mean)},$$

$$\Sigma_t^{(N)} := \frac{1}{N-1}\sum_{i=1}^N (X_t^i - m_t^{(N)})(X_t^i - m_t^{(N)})^\top \text{ (empirical covariance)}$$

Objectives:

- Convergence $m_t^{(N)} \to m_t, \quad \Sigma_t^{(N)} \to \Sigma_t$
- Convergence of the empirical distribution

Stochastic linear FPF:

$$\mathrm{d}X_t^i = AX_t^i\,\mathrm{d}t + \sigma_B\,\mathrm{d}B_t^i + \mathsf{K}_t^{(N)}(\mathrm{d}Z_t - \frac{HX_t^i + Hm_t^{(N)}}{2}\,\mathrm{d}t), \quad X_0^i \overset{\text{i.i.d}}{\sim} p_0$$

for $i = 1, \ldots, N$

Deterministic linear FPF:

$$\mathrm{d}X_t^i = AX_t^i\,\mathrm{d}t + \frac{1}{2}\sigma_B\sigma_B^\top {\Sigma_t^{(N)}}^{-1}(X_t^i - m_t^{(N)})\,\mathrm{d}t + \mathsf{K}_t^{(N)}(Z_t - \frac{HX_t + Hm_t^{(N)}}{2}\,\mathrm{d}t)$$

where the mean-field terms are empirically approximated

$$m_t^{(N)} := \frac{1}{N}\sum_{i=1}^N X_t^i \text{ (empirical mean)},$$

$$\Sigma_t^{(N)} := \frac{1}{N-1}\sum_{i=1}^N (X_t^i - m_t^{(N)})(X_t^i - m_t^{(N)})^\top \text{ (empirical covariance)}$$

Objectives:

- Convergence $m_t^{(N)} \to m_t, \quad \Sigma_t^{(N)} \to \Sigma_t$
- Convergence of the empirical distribution

Stochastic linear FPF:

$$\mathrm{d}X_t^i = AX_t^i \,\mathrm{d}t + \sigma_B \,\mathrm{d}B_t^i + \mathsf{K}_t^{(N)}(\mathrm{d}Z_t - \frac{HX_t^i + Hm_t^{(N)}}{2}\,\mathrm{d}t), \quad X_0^i \overset{\text{i.i.d}}{\sim} p_0$$

for $i = 1, \ldots, N$

Deterministic linear FPF:

$$\mathrm{d}X_t^i = AX_t^i \,\mathrm{d}t + \frac{1}{2}\sigma_B\sigma_B^\top \Sigma_t^{(N)^{-1}}(X_t^i - m_t^{(N)})\,\mathrm{d}t + \mathsf{K}_t^{(N)}(Z_t - \frac{HX_t + Hm_t^{(N)}}{2}\,\mathrm{d}t)$$

where the mean-field terms are empirically approximated

$$m_t^{(N)} := \frac{1}{N}\sum_{i=1}^N X_t^i \text{ (empirical mean)},$$

$$\Sigma_t^{(N)} := \frac{1}{N-1}\sum_{i=1}^N (X_t^i - m_t^{(N)})(X_t^i - m_t^{(N)})^\top \text{ (empirical covariance)}$$

Objectives:

- Convergence $m_t^{(N)} \to m_t, \quad \Sigma_t^{(N)} \to \Sigma_t$
- Convergence of the empirical distribution

Three formulations for linear FPF and EnKF:

$$\mathrm{d}X_t = AX_t + \gamma_1 \sigma_B \,\mathrm{d}B_t + \frac{1-\gamma_1^2}{2}\sigma_B\sigma_B^\top \Sigma_t^{-1}(X_t - m_t)\,\mathrm{d}t$$
$$+ \mathsf{K}_t^{(N)}\Big(Z_t - \frac{(1+\gamma_2^2)HX_t + (1-\gamma_2^2)Hm_t^{(N)}}{2}\,\mathrm{d}t + \gamma_2\,\mathrm{d}W_t\Big)$$

- $\gamma_1 = 1, \gamma_2 = 1$: EnKF with perturbed observation: [Bergemann, et. al. 2012]
- $\gamma_1 = 1, \gamma_2 = 0$: Square root EnKF [Bergemann, et. al. 2012] [Yang, et. al. 2013]
- $\gamma_1 = 0, \gamma_2 = 0$: Deterministic linear FPF [Taghvaei, et. al. 2016]

Error analysis of the EnKF

- Discrete time:[Le Gland, 2009] [Mandel, 2011][Kwiatkowski, 2015] [Kelly, 2014]
- Continuous time: [Del Moral, 2016, 2017] [Bishop, 2018][De Wiljes, 2016]

Current result: Uniform in time $O(\frac{1}{\sqrt{N}})$ convergence under stability and full observation assumption

Assumptions:

- The system (A, H) is detectable and (A, σ_B) is stabilizable

Stability of the Kalman filter:

- There exists a unique solution Σ_∞ to ARE,
- The error covariance converges exponentially fast

$$\lim_{t\to\infty} e^{2\lambda t}\|\Sigma_t - \Sigma_\infty\| = 0$$

- Starting from two initial conditions (m_0, Σ_0) and $(\tilde{m}_0, \tilde{\Sigma}_0)$ the means converge exponentially fast

$$\lim_{t\to\infty} e^{2\lambda t}\mathsf{E}[\|m_t - \tilde{m}_t\|] = 0$$

D. Ocone and E. Pardoux. Asymptotic stability of the optimal filter with respect to its initial condition. SIAM, 1996.

Error analysis of the deterministic linear FPF

Evolution of mean and covariance:

$$\mathrm{d}m_t^{(N)} = \underbrace{Am_t^{(N)}\,\mathrm{d}t + \mathsf{K}_t^{(N)}(\,\mathrm{d}Z_t - Hm_t^{(N)}\,\mathrm{d}t)}_{\text{Kalman filter}}$$

$$\mathrm{d}\Sigma_t^{(N)} = \underbrace{\mathsf{Ric}(\Sigma_t^{(N)})\,\mathrm{d}t}_{\text{Kalman filter}}$$

Assumptions:

- The system (A, H) is detectable and (A, σ_B) is stabilizable
- $\Sigma_0^{(N)}$ is invertible

Proposition

Under assumptions (I) and (II) there exists $\lambda_0 > 0$ such that:

$$\mathsf{E}[|m_t^{(N)} - m_t|^2] \le (\text{const.})\frac{e^{-2\lambda_0 t}}{N}$$

$$\mathsf{E}[\|\Sigma_t^{(N)} - \Sigma_t\|_F^2] \le (\text{const.})\frac{e^{-4\lambda_0 t}}{N}$$

Evolution of mean and covariance:

$$\mathrm{d}m_t^{(N)} = \underbrace{Am_t^{(N)}\,\mathrm{d}t + \mathsf{K}_t^{(N)}(\,\mathrm{d}Z_t - Hm_t^{(N)}\,\mathrm{d}t)}_{\text{Kalman filter}}$$

$$\mathrm{d}\Sigma_t^{(N)} = \underbrace{\mathsf{Ric}(\Sigma_t^{(N)})\,\mathrm{d}t}_{\text{Kalman filter}}$$

Assumptions:

- The system (A, H) is detectable and (A, σ_B) is stabilizable
- $\Sigma_0^{(N)}$ is invertible

Proposition

Under assumptions (I) and (II) there exists $\lambda_0 > 0$ such that:

$$\mathsf{E}[|m_t^{(N)} - m_t|^2] \leq (\text{const.})\frac{e^{-2\lambda_0 t}}{N}$$

$$\mathsf{E}[\|\Sigma_t^{(N)} - \Sigma_t\|_F^2] \leq (\text{const.})\frac{e^{-4\lambda_0 t}}{N}$$

Evolution of mean and covariance:

$$\mathrm{d}m_t^{(N)} = \underbrace{Am_t^{(N)}\,\mathrm{d}t + \mathsf{K}_t^{(N)}(\,\mathrm{d}Z_t - Hm_t^{(N)}\,\mathrm{d}t)}_{\text{Kalman filter}} + \underbrace{\frac{\sigma_B}{\sqrt{N}}\,\mathrm{d}\tilde{B}_t}_{\text{stochastic term}}$$

$$\mathrm{d}\Sigma_t^{(N)} = \underbrace{\mathsf{Ric}(\Sigma_t^{(N)})\,\mathrm{d}t}_{\text{Kalman filter}} + \underbrace{\frac{\mathrm{d}M_t}{\sqrt{N}}}_{\text{stochastic term}}$$

Remark: Stochastic terms scale as $O(\frac{1}{\sqrt{N}})$

Current result: Scalar case

$$\mathsf{E}[|\Sigma_t^{(N)} - \Sigma_t|^2] \leq (\text{const.})\,\frac{e^{-2\frac{\sigma_B^2}{\Sigma_\infty}t}}{N} + \frac{(\text{const.})}{N}$$

EnKF Literature: Uniform converegnce under stronger assumptions: system is stable and fully observable

Evolution of mean and covariance:

$$\mathrm{d}m_t^{(N)} = \underbrace{Am_t^{(N)}\,\mathrm{d}t + \mathsf{K}_t^{(N)}(\,\mathrm{d}Z_t - Hm_t^{(N)}\,\mathrm{d}t)}_{\text{Kalman filter}} + \underbrace{\frac{\sigma_B}{\sqrt{N}}\,\mathrm{d}\tilde{B}_t}_{\text{stochastic term}}$$

$$\mathrm{d}\Sigma_t^{(N)} = \underbrace{\mathsf{Ric}(\Sigma_t^{(N)})\,\mathrm{d}t}_{\text{Kalman filter}} + \underbrace{\frac{\mathrm{d}M_t}{\sqrt{N}}}_{\text{stochastic term}}$$

Remark: Stochastic terms scale as $O(\frac{1}{\sqrt{N}})$

Current result: Scalar case

$$\mathsf{E}[|\Sigma_t^{(N)} - \Sigma_t|^2] \leq (\text{const.})\ \frac{e^{-2\frac{\sigma_B^2}{\Sigma_\infty}t}}{N} + \frac{(\text{const.})}{N}$$

EnKF Literature: Uniform converegnce under stronger assumptions: system is stable and fully observable

Evolution of mean and covariance:

$$\mathrm{d}m_t^{(N)} = \underbrace{Am_t^{(N)}\,\mathrm{d}t + \mathsf{K}_t^{(N)}(\,\mathrm{d}Z_t - Hm_t^{(N)}\,\mathrm{d}t)}_{\text{Kalman filter}} + \underbrace{\frac{\sigma_B}{\sqrt{N}}\,\mathrm{d}\tilde{B}_t}_{\text{stochastic term}}$$

$$\mathrm{d}\Sigma_t^{(N)} = \underbrace{\mathsf{Ric}(\Sigma_t^{(N)})\,\mathrm{d}t}_{\text{Kalman filter}} + \underbrace{\frac{\mathrm{d}M_t}{\sqrt{N}}}_{\text{stochastic term}}$$

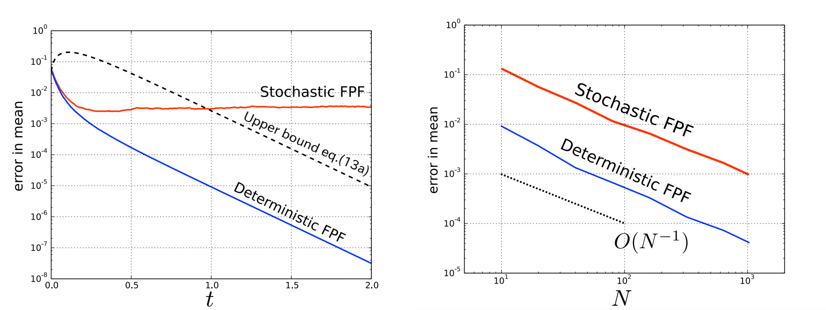
Remark: Stochastic terms scale as $O(\frac{1}{\sqrt{N}})$

Current result: Scalar case

$$\mathsf{E}[|\Sigma_t^{(N)} - \Sigma_t|^2] \leq \text{(const.)}\ \frac{e^{-2\frac{\sigma_B^2}{\Sigma_\infty}t}}{N} + \frac{\text{(const.)}}{N}$$

EnKF Literature: Uniform converegnce under stronger assumptions: system is stable and fully observable

Numerics



error in mean
Stochastic FPF
Upper bound eq.(13a)
Deterministic FPF
t
error in mean
Stochastic FPF
Deterministic FPF
$O(N^{-1})$
N

Approach: Construct independent copies of the mean-field process coupled to particles

$$\mathrm{d}\bar{X}^i_t = A\bar{X}^i_t\,\mathrm{d}t + \frac{1}{2}\sigma_B\sigma_B^\top\bar{\Sigma}_t^{-1}(X^i_t - \bar{m}_t)\,\mathrm{d}t + \bar{\mathsf{K}}_t(Z_t - \frac{HX_t + H\bar{m}_t}{2}\,\mathrm{d}t)$$

$$\mathrm{d}X^i_t = AX^i_t\,\mathrm{d}t + \frac{1}{2}\sigma_B\sigma_B^\top{\Sigma_t^{(N)}}^{-1}(X^i_t - m^{(N)}_t)\,\mathrm{d}t + \mathsf{K}^{(N)}_t(Z_t - \frac{HX_t + Hm^{(N)}_t}{2}\,\mathrm{d}t)$$

with $X^i_0 = \bar{X}^i_0$, for $i = 1, \ldots, N$

Proposition

Consider the deterministic linear FPF for the scalar case. Then, under assumptions (I) and (II),

$$\mathsf{E}[|X^i_t - \bar{X}^i_t|^2] \leq \frac{\text{(const)}}{N}$$

$$\mathsf{E}[\Big|\frac{1}{N}\sum_{i=1}^N f(X^i_t) - \mathsf{E}[f(X_t)|\mathcal{Z}_t]\Big|^2] \leq \frac{\text{(const)}}{N}, \quad \forall f \in C_b(\mathbb{R}^d)$$

A. Sznitman. Topics in propagation of chaos, 1991

Conclusion

This work:

- Uniform convergence of mean and covariance for the deterministic linear FPF
- Uniform convergence of the empirical distribution for the deterministic linear FPF (for the scalar case)

Future work:

- Quantify the dependence of the error bounds on the dimension
- Prove or disprove uniform convergence for the stochastic linear FPF under the detectable and stabilizable assumpptions (open problem for the vector case)

Thank you for your attention!

Conclusion

This work:

- Uniform convergence of mean and covariance for the deterministic linear FPF
- Uniform convergence of the empirical distribution for the deterministic linear FPF (for the scalar case)

Future work:

- Quantify the dependence of the error bounds on the dimension
- Prove or disprove uniform convergence for the stochastic linear FPF under the detectable and stabilizable assumpptions (open problem for the vector case)

Thank you for your attention!