Gain Function Approximation in the Feedback Particle Filter

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Joint work with P. G. Mehta

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Dec 14, 2016

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Gain Function Approximation in FPF
Problem formulation

Poisson equation:
$$-\frac{1}{\rho(x)}\nabla\cdot(\rho(x)\nabla\phi(x)) = h(x)-\hat{h}$$
$$\int_{\mathbb{R}^d}\phi(x)\rho(x)\,\mathrm{d}x = 0$$

- $\rho:\mathbb{R}^d\to\mathbb{R}^+$ (prob. density)
- $h:\mathbb{R}^d\to\mathbb{R}$ (given function),
- $\hat{h} := \int h(x)\rho(x)\,\mathrm{d}x$
- $\phi:\mathbb{R}^d\to\mathbb{R}$ (solution)

Problem:

Given: $\{X^1,\ldots,X^N\}\overset{\text{i.i.d}}{\sim}\rho$

Find: $\{\nabla\phi(X^1),\ldots,\nabla\phi(X^N)\}$ (approximately)

Almost like a statistical learning problem

R. S. Laugesen, P. G. Mehta, S. P. Meyn, and M. Raginsky. Poisson Equation in Nonlinear Filtering. SICON, 2015

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Feedback Particle Filter

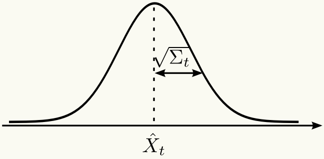
Generalization of the Kalman Filter

Kalman Filter:

$$\mathrm{d}X_t = AX_t\,\mathrm{d}t + \mathrm{d}B_t$$
$$\mathrm{d}Z_t = HX_t\,\mathrm{d}t + \mathrm{d}W_t$$

$\mathsf{P}(X_t|\mathcal{Z}_t)$ = Gaussian $N(\hat{X}_t, \Sigma_t)$,

$$\mathrm{d}\hat{X}_t = A\hat{X}_t\,\mathrm{d}t + \mathsf{K}_t(\,\mathrm{d}Z_t - H\hat{X}_t\,\mathrm{d}t)$$
$$\frac{\mathrm{d}\Sigma_t}{\mathrm{d}t} = \ldots \text{(Riccati equation)}$$



Feedback Particle Filter:

$$\mathrm{d}X_t = a(X_t)\,\mathrm{d}t + \mathrm{d}B_t$$
$$\mathrm{d}Z_t = h(X_t)\,\mathrm{d}t + \mathrm{d}W_t$$

$\mathsf{P}(X_t|\mathcal{Z}_t) \approx$ empirical dist. $\{X^1, \ldots, X^N\}$,

$$\mathrm{d}X^i_t = a(X^i_t)\,\mathrm{d}t + \mathrm{d}B^i_t + \mathsf{K}_t(X^i_t) \circ (\,\mathrm{d}Z_t - \frac{h(X^i_t) + \hat{h}_t}{2}\,\mathrm{d}t)$$

Challenge: Compute the gain function $\mathsf{K}_t := \nabla\phi$ from Poisson eq.

T. Yang, P. G. Mehta, and S. P. Meyn. feedback particle filter, *TAC*, 2013
T. Yang, R. S. Laugesen, P. G. Mehta, and S. P. Meyn. Multivariable feedback particle filter, *Automatica*, 2016

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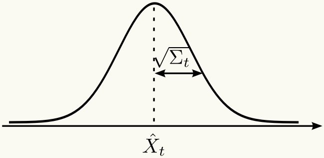

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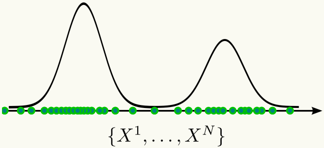

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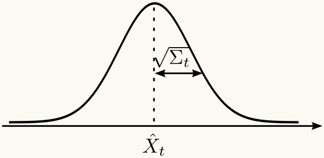

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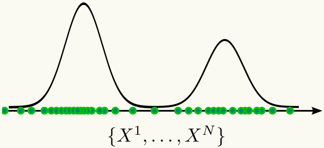

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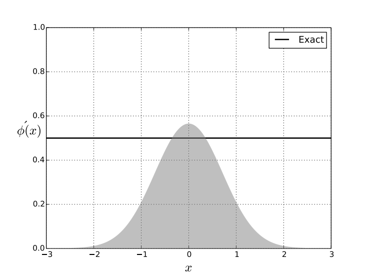
Gaussian distribution
Linear observation



$\mathsf{K}_t(x) = \text{constant} \quad \text{(Kalman gain)}$

Non-Gaussian distribution
Nonlinear observation

$\mathsf{K}_t(x) = \ldots \text{(Nonlinear gain)}$

Gaussian distribution
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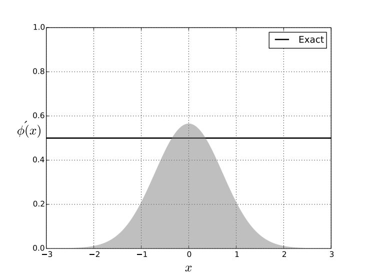

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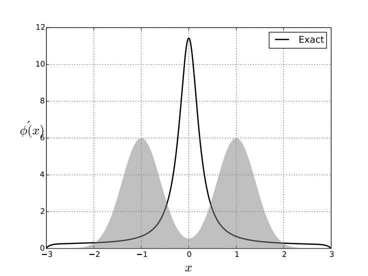

$\mathsf{K}_t(x) = \ldots$ (Nonlinear gain)

Poisson equation: $$-\frac{1}{\rho}\nabla\cdot(\rho\nabla\phi) = h - \hat{h}$$

Weighted Laplacian: $$\Delta_\rho\phi := \frac{1}{\rho}\nabla\cdot(\rho\nabla\phi) = \Delta\phi + \nabla\log\rho\cdot\nabla\phi$$

PDE

- Markov Diffusion operators [D. Bakry, et. al. 2013]
- Heat kernels [A. Grigoryan, 2009]
- Optimal transportation [C. Villani, 2003]

Stochastic analysis

- Simulation and optimization theory for Markov models [S. Meyn, R. Tweedie, 2012]

Statistical learning

- Nonlinear dimensionality reduction [M. Belkin, 2003]
- Diffusion maps [R. Coifman, S. Lafon, 2006]
- Spectral clustering [M. Hein, et. al. 2006]

FPF (theory and application):

- T. Yang, et. al. Automatica, 2015.
- K. Berntorp, Fusion, 2015.
- P. M. Stano, et. al., 2014.

} (Constant gain and Galerkin approximation)

Gain function approximation:

- K. Berntorp, P. Grover, ACC, 2016. (Data driven approach based on POD)
- Y. Matsuura, et. al. 2016. (Continuation method)
- A. Radhakrishnan, A. Devraj, and S. Meyn. CDC, 2016. (TD learning)

Also in other nonlinear filtering algorithms

- Particle flow filter [F. Daum, J. Huang, 2010]
- Approxiamte representation of SPDE [D. Crisan, J. Xiong, 2005]
- Dynamical systems framework for intermittent data assimilation [S. Riech 2011]
- Continuous-discrete time FPF [T. Yang, et. al. 2014]

Different Formulations of the Poisson Equation

Weak formulation: (Galerkin)

$$\langle \nabla\phi, \nabla\psi \rangle = \langle h - \hat{h}, \psi \rangle, \quad \forall \psi \in H^1(\mathbb{R}^d, \rho)$$

where $\langle f, g \rangle := \int f(x)g(x)\rho(x)\,\mathrm{d}x$

Semigroup formulation: (kernel-based)

$$\phi = P\phi + \tilde{h}$$

where $P := e^{\epsilon\Delta_\rho}$ and $\tilde{h} := \int_0^t e^{s\Delta_\rho}(h - \hat{h})\,\mathrm{d}s$

Variational formulation: (Neural net ?)

$$\min_{\phi \in H_0^1(\mathbb{R}^d, \rho)} \mathsf{E}\left[\frac{1}{2}|\nabla\phi(X)|^2 - \phi(X)(h(X) - \hat{h})\right]$$

where $X \sim \rho$

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Strong form:

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Galerkin approximation:

$$\langle \nabla\phi^{(M)}, \nabla\psi\rangle = \langle h - \hat{h}, \psi\rangle, \quad \forall \psi \in S$$

where $S = \text{span}\{\psi_1, \ldots, \psi_M\}$

Empirical approximation

$$\frac{1}{N}\sum_{i=1}^{N}\nabla\phi^{(M)}(X^i)\cdot\nabla\psi(X^i) = \frac{1}{N}\sum_{i=1}^{N}(h(X^i) - \hat{h})\psi(X^i), \quad \forall \psi \in S$$

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- Select basis functions $\{\psi_1, \ldots, \psi_M\}$
- Express the approximate solution as

$$\phi^{(M,N)}(x) = \sum_{m=1}^{M} c_m \psi_m(x)$$

- Obtain $c = (c_1, \ldots, c_M)$ by solving

$$Ac = b$$

where

$$A_{ml} = \langle \nabla\psi_m, \nabla\psi_l \rangle \approx \frac{1}{N} \sum_{i=1}^{N} \nabla\psi_m(X^i) \cdot \nabla\psi_l(X^i)$$

$$b_m = \langle \psi_m, h \rangle \approx \frac{1}{N} \sum_{i=1}^{N} \psi_m(X^i) h(X^i) - \hat{h})$$

Special case: The basis functions are eigenfunctions of Δ_ρ

$$\mathsf{E}\left[\|\nabla\phi - \nabla\phi^{(M,N)}\|_{L^2}\right] \le \underbrace{\frac{1}{\sqrt{\lambda_M}}\|h - \Pi_S h\|_{L^2}}_{\text{Bias}} + \underbrace{\frac{1}{\sqrt{N}}\|h\|_\infty \sqrt{\sum_{m=1}^{M}\frac{1}{\lambda_m}}}_{\text{Variance}}$$

- ϕ: exact solution
- $\phi^{(M,N)}$: approximate solution
- $\{\lambda_m\}_{m=1}^\infty$ the eigenvalues

It is a projection:

$$\nabla\phi^{(M)} = \arg\min_{\nabla\psi\in S} \|\nabla\phi - \nabla\psi\|_2$$

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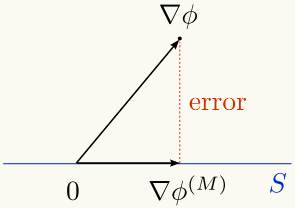
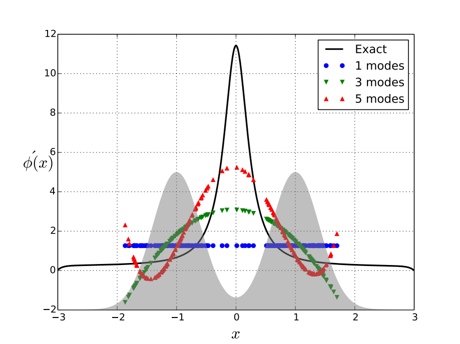


Galerkin Approximation
Numerical result



Issues:

- Choice of basis functions
- Singularity of A
- Computationally scales with $O(Nd^p)$

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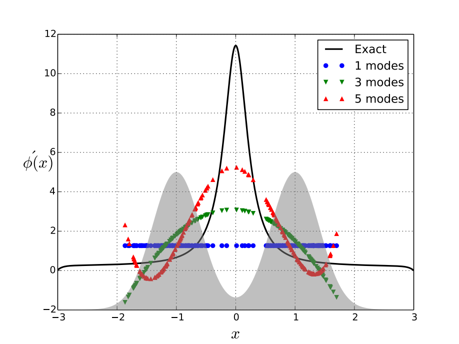

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Semigroup identity: $e^{\epsilon \Delta_\rho} = I + \int_0^\epsilon e^{s\Delta_\rho} \Delta_\rho \, \mathrm{d}s$

Semigroup formulation:

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where $\tilde{h} := \int_0^\epsilon e^{s\Delta_\rho} (h - \hat{h}) \, \mathrm{d}s$

Kernel representation:

$$\phi(x) = \int \tilde{k}_\epsilon(x,y) \phi(y) \rho(y) \, \mathrm{d}y + \tilde{h}(x)$$

Empirical approximation:

$$\phi(x) = \frac{1}{N} \sum_{i=1}^{N} \tilde{k}_\epsilon(x, X^i) \phi(X^i) + \tilde{h}(x)$$

But $\tilde{k}_\epsilon(x,y) = ?$

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$$\phi(x) = \frac{1}{N}\sum_{i=1}^{N} \tilde{k}_\epsilon(x, X^i)\phi(X^i) + \tilde{h}(x)$$

But $\tilde{k}_\epsilon(x,y) = ?$

Poisson equation: $-\Delta_\rho \phi = h - \hat{h}$

Semigroup identity: $e^{\epsilon \Delta_\rho} = I + \int_0^\epsilon e^{s\Delta_\rho} \Delta_\rho \, \mathrm{d}s$

Semigroup formulation:

$$\phi = e^{\epsilon \Delta_\rho} \phi + \tilde{h}$$

where $\tilde{h} := \int_0^\epsilon e^{s\Delta_\rho} (h - \hat{h}) \, \mathrm{d}s$

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Kernel-based Approximation

Special case: $\rho = 1$

$$e^{\epsilon\Delta} f(x) = \int g_\epsilon(x,y) f(y) \, \mathrm{d}y. \quad \text{(for all } \epsilon > 0)$$

where g_ϵ is the Gaussian kernel.

In general:

$$e^{\epsilon\Delta_\rho} f(x) \approx \int \frac{1}{n_\epsilon(x)} \frac{g_\epsilon(x,y)}{\sqrt{\int g_\epsilon(y,z)\rho(z)\,\mathrm{d}z}} f(y)\rho(y)\,\mathrm{d}y := T_\epsilon f(x) \quad \text{(for } \epsilon \downarrow 0)$$

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Exact solution:

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Approximation:

$$\phi_\epsilon^{(N)}(x) := T_\epsilon^{(N)}\phi_\epsilon^{(N)}(x) + \epsilon(h(x) - \hat{h}),$$

Numerics:

$$\Phi = \mathbf{T}\Phi + \epsilon(\mathbf{h} - \hat{h})$$

- $\Phi = (\Phi_\epsilon^{(N)}(X^1), \ldots, \Phi_\epsilon^{(N)}(X^N))$
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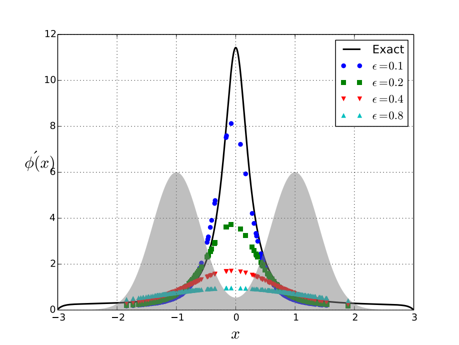
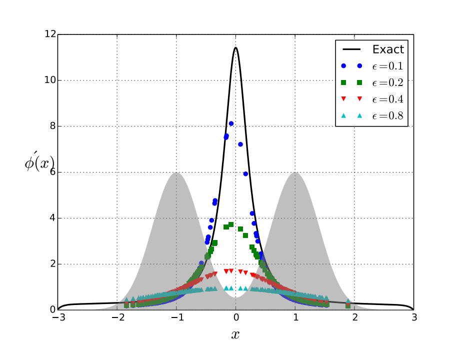


Properties

1. No singularity
2. Easy extension to Manifolds [C. Zhang, et. al. CDC 2015]
3. Better error bounds
4. Computational cost $O(N^2)$ (good in high dimensions)

Kernel-based approximation
Numerical result



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Kernel-based approximation
Error Analysis

$$\phi_\epsilon^{(N)} \xrightarrow[\text{Variance}]{N \uparrow \infty} \phi_\epsilon \xrightarrow[\text{Bias}]{\epsilon \downarrow 0} \phi$$

(Exact solution) $\phi(x) = e^{\epsilon\Delta_\rho}\phi(x) + \tilde{h}(x)$

(Kernel approx.) $\phi_\epsilon(x) = T_\epsilon \phi_\epsilon(x) + \epsilon(h(x) - \hat{h})$

(Empirical approx.) $\phi_\epsilon^{(N)}(x) = T_\epsilon^{(N)} \phi_\epsilon^{(N)}(x) + \epsilon(h(x) - \hat{h})$

Error bound:

$$\mathsf{E}\left[\|\nabla\phi - \nabla\phi_\epsilon^{(N)}\|_2\right] \leq \underbrace{O(\epsilon)}_{\text{Bias}} + \underbrace{O\left(\frac{1}{\epsilon^{1+d/4}\sqrt{N}}\right)}_{\text{Variance}}$$

on bounded domain.

Future work: Extension to unbounded domain

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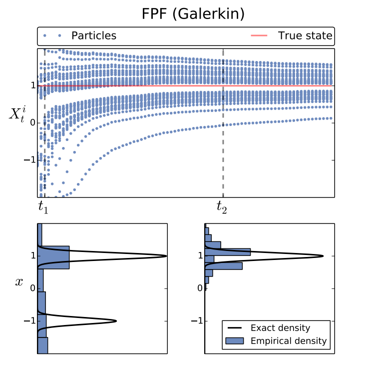
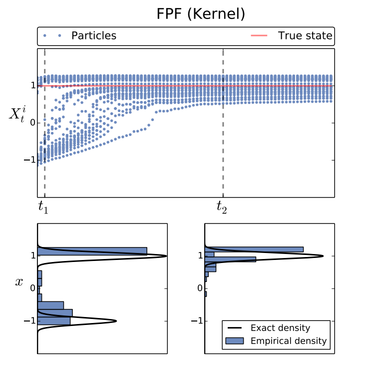
Numerical result
Nonlinear filtering example

$$\mathrm{d}X_t = 0, \quad X_0 \sim \frac{1}{2}N(-1,\sigma^2) + \frac{1}{2}N(+1,\sigma^2)$$
$$\mathrm{d}Z_t = X_t\,\mathrm{d}t + \sigma_w\,\mathrm{d}W_t$$





Final slide

Thank you!