Mean-field Control for Machine Learning

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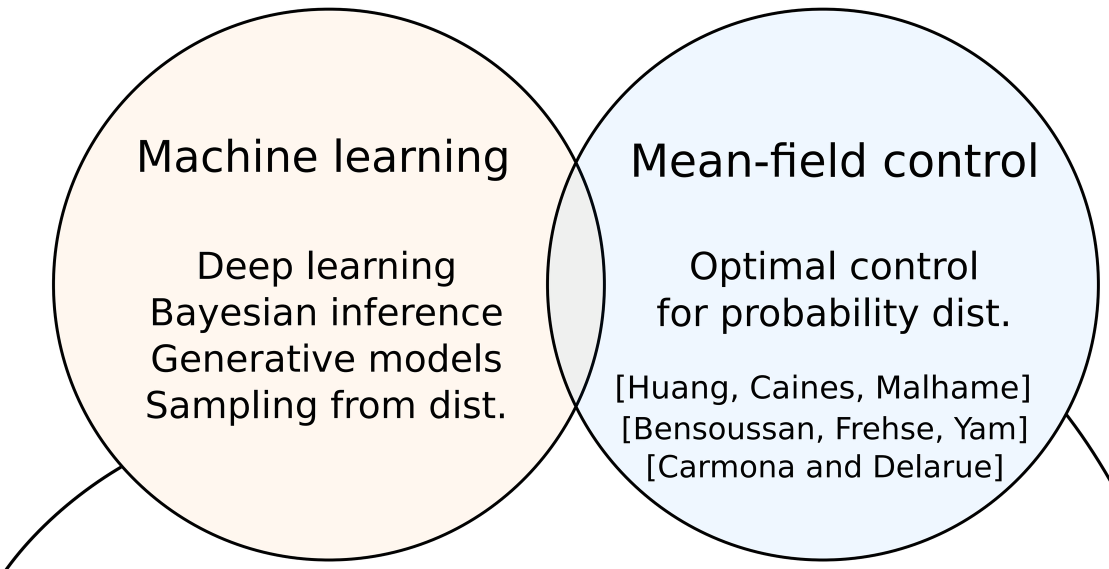
Research overview



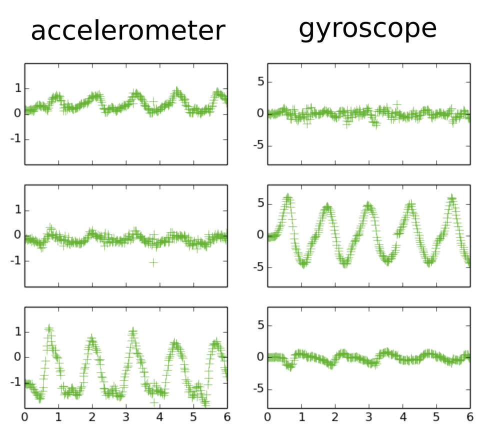
Many machine learning problems can be formulated as an optimization problem on the space of probability distributions

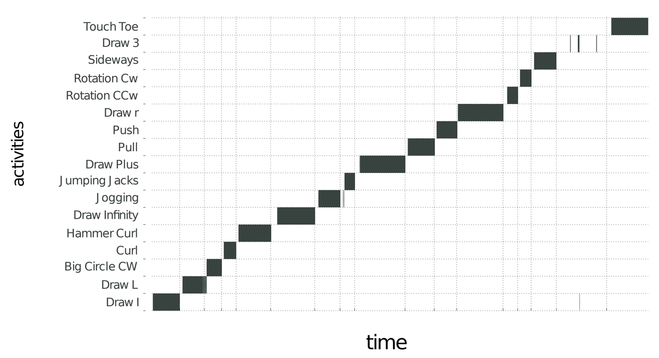
My research interest: Mean-field control can be used to design algorithms realized as a system of controlled interacting particles to solve ML problems

- **A. Taghvaei**, P. G. Mehta. *Accelerated gradient flow for probability distributions* (under review, ICLR'19)
- **A. Taghvaei**, J. Kim, P. G. Mehta. *How regularization affects the critical points in linear neural networks* (NIPS'17)
- **A. Taghvaei**, J de Wiljes, P. G. Mehta, and S. Reich. Kalman filter and its modern extensions for the continuous-time nonlinear filtering problem. ASME, Nov, 2017
- C. Zhang, **A. Taghvaei**, P. G. Mehta. *A mean-field optimal control formulation for global optimization*, (TAC), May, 2018

Nonlinear filtering and feedback particle filter

Filtering problem:





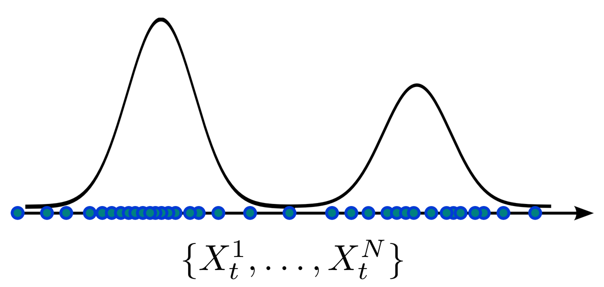
State X_t Observation Z_t Detection $\mathsf{P}(X_t|\mathcal{Z}_t)$

Feedback Particle Filter:

$$\mathrm{d}X_t^i = \underbrace{u_t(X_t^1,\ldots,X_t^N)}_{\text{control law}}\,\mathrm{d}t + \underbrace{\mathsf{K}_t(X_t^1,\ldots,X_t^N)}_{\text{control law}}\,\mathrm{d}Z_t, \quad \text{for} \quad i=1,\ldots,N$$

Choose the control law such that the empirical distribution of the particles approximates the posterior distribution

$$\frac{1}{N}\sum_{i=1}^N \delta_{X_t^i} \approx \mathsf{P}(X_t|\mathcal{Z}_t)$$



Questions:

1. How to design the control law?
2. How to compute the control law?
3. What is the total error of the algorithm?

T. Yang, R. S. Laugesen, P. G. Mehta, and S. P. Meyn. Multivariable feedback particle filter, *Automatica*, 2015

(2) Gain function approximation

Control law from [T. Yang, et. al.] :

$$\mathrm{d}\bar{X}_t = \text{(dynamics)} + \underbrace{\mathsf{K}_t(\bar{X}_t)\,\mathrm{d}\bar{I}_t}_{\text{correction update}}$$

Gain function $\mathsf{K}_t(x) = \nabla\phi(x)$ where ϕ solves the Poisson eq.

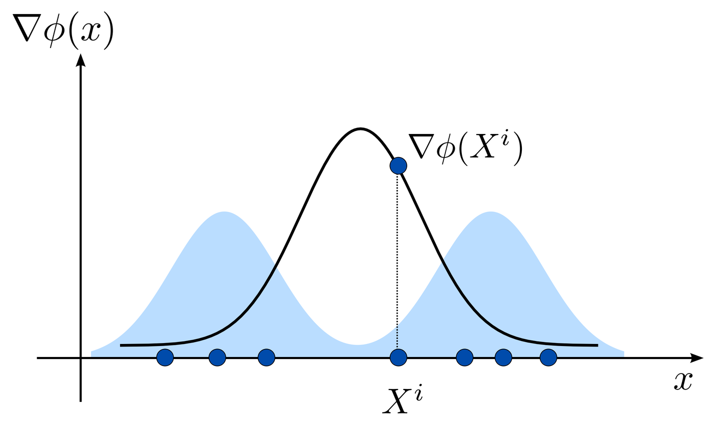
Poisson equation:

$$-\frac{1}{\rho(x)}\nabla\cdot(\rho(x)\nabla\phi(x)) = h(x)$$

- ρ is a prob. density
- h is a given function,

Given: $\{X^1,\ldots,X^N\} \overset{\text{i.i.d}}{\sim} \rho$

Find: $\{\nabla\phi(X^1),\ldots,\nabla\phi(X^N)\}$



Constant gain function approximation and relation to ensemble the Kalman filter

Constant gain approximation:

$$\mathsf{K}_{\text{const.}} = \int (h(x)-\hat{h})\rho(x)\,\mathrm{d}x \approx \frac{1}{N}\sum_{i=1}^N (h(X^i)-\hat{h}^{(N)})X^i$$

$$\boxed{\text{FPF} \xrightarrow{\text{const. gain}} \text{EnKf}}$$

- EnKf is widely used in geophysical applications where the state dimension is large as an alternative to the Kalman filter [Evensen, 1994]
- EnKf is not exact for nonlinear and non Gaussian systems
- FPF is the generalization of the EnKf for non-linear and non-Gaussian systems

A. Taghvaei, J de Wiljes, P. G. Mehta, and S. Reich. Kalman filter and its modern extensions for the continuous-time nonlinear filtering problem. ASME, 2017

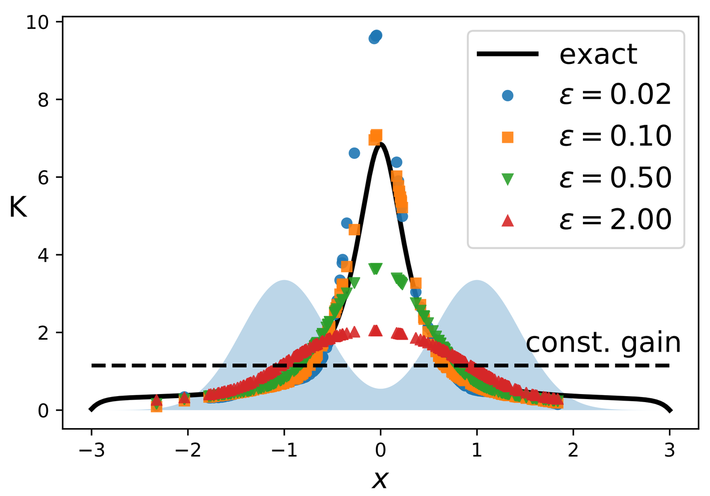
Diffusion map approximation of the gain

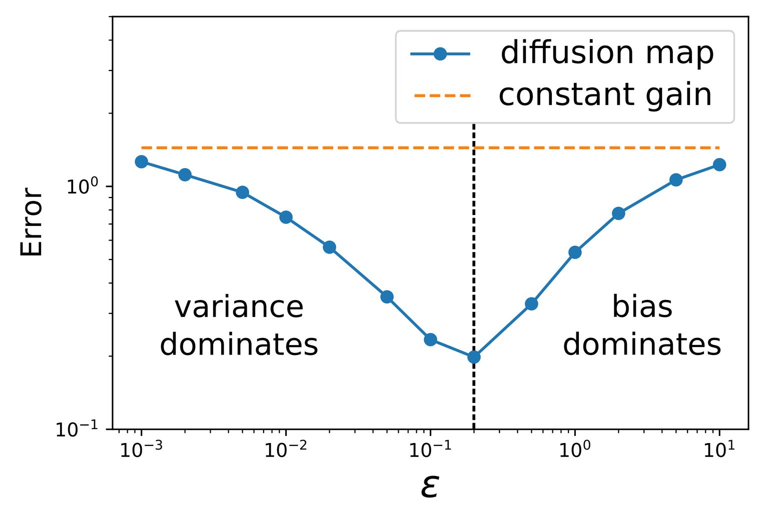
- **Stochastic formulation:**

$$\phi = P_\epsilon\phi + \int_0^\epsilon P_s(h-\hat{h})\,\mathrm{d}s$$

where $\{P_t\}$ is the semigroup for $\Delta_\rho := \frac{1}{\rho}\nabla\cdot(\rho\nabla)$

- Approximate P with a Markov matrix using particles [Coifman (2006)]





Error analysis:

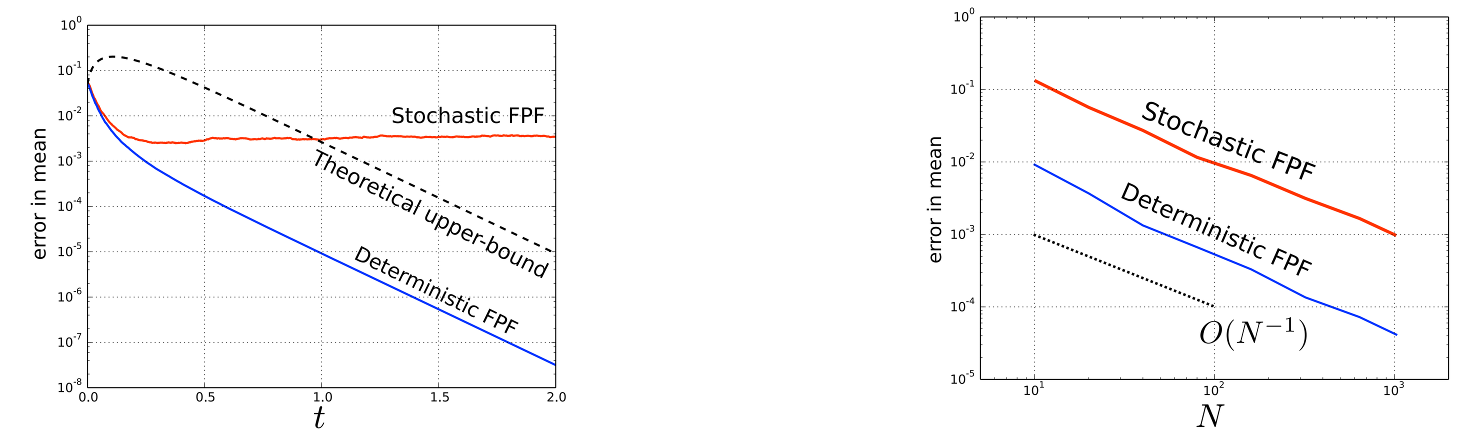
$$\text{Total error} \le \underbrace{O(\epsilon)}_{\text{Bias}} + \underbrace{O\Big(\frac{1}{\epsilon^{1+d/2}\sqrt{N}}\Big)}_{\text{Variance}}$$

A. Taghvaei, P. G. Mehta, Gain function approximation in the feedback particle filter, CDC, 2016
A. Taghvaei, P. G. Mehta. S. P. Meyn, Error Estimates for the Kernel Gain Function Approximation in the Feedback Particle Filter, ACC, 2017

(3) Error analysis of the linear FPF

Question: Convergence of the empirical distribution to the mean-field distribution. Upper-bound for mean squared error in estimation

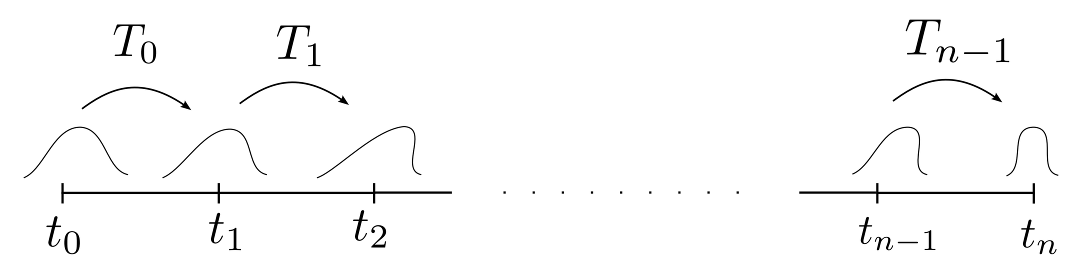
Assumption: Linear Gaussian model



A. Taghvaei, P. G. Mehta, Error analysis of the linear FPF, (ACC), 2018
A. Taghvaei, P. G. Mehta, Error analysis of the stochastic linear FPF, (CDC), 2018

(1) Optimal transport formulation of FPF

Question: How to design the control law in the FPF
Idea: View filtering as a transportation problem from prior distribution to the posterior distribution. Form a unique control law from optimal transport maps



A. Taghvaei, P. G. Mehta. An Optimal Transport Formulation of Linear Feedback Particle Filter, (ACC), 2016

Other approaches: Duality, Entropy minimization, Schrödinger bridge

Summary

(1) Design: Formulate the problem in the space of probability distributions using the mean-field approach. Use variational principles to model the objective. Use optimal transportation theory and mean-field optimal control to obtain the mean-field control law

(2) Approximation: Compute the mean-field control law in terms of finite number of particles. Use tools from stochastic approximation and statistical learning to design and analyze algorithms

(3) Error analysis: Study the total error of the algorithm and the convergence of the finite-N system to the mean-field limit.

$$\text{Mean-field sys.} \underset{\text{Convergence}}{\overset{\text{Approximation}}{\rightleftarrows}} \text{Finite-N sys.}$$

Internship experiences

1) AI researcher , with Dr. Amin Jalali, Technicolor AI research lab, Palo Alto, Summer, 2018
Project: Restricted Convex Potentials for Approximating the Wasserstein Metric and the Optimal Transport Mapping

2) Algorithm Developer , with university start-up, Rithmio, 2014-2015
Project: Development of learning algorithms for real time classification of physical activities, based on wearable inertial sensors

Other activities

- Organizer of the CSL student conference, UIUC, 2015, 2016, 2018
- Mentorship of five undergraduate students

Acknowledgement

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