Error Analysis of the Stochastic Linear Feedback Particle Filter

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Outline

- Filtering problem in linear Gaussian setting
- Kalman filter (1960s)

  Kalman filter is exact, but it is computationally expensive for high dim. problems
- Ensemble Kalman filter (1990s) linear Feedback Particle Filter (2010s)

  They are computationally efficient, but have approximation errors
- Error analysis of the FPF and EnKF (2017-18)

  If the system is stable and fully observable, then uniform error bounds are guaranteed

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Filtering problem: Linear Gaussian setting

Model:

$$\text{State process:} \quad \mathrm{d}X_t = AX_t\,\mathrm{d}t + \sigma_B\,\mathrm{d}B_t, \quad \text{(linear dynamics)}$$

$$\text{Observation process:} \quad \mathrm{d}Z_t = HX_t\,\mathrm{d}t + \mathrm{d}W_t, \quad \text{(linear observation)}$$

$$\text{Prior distribution:} \quad X_0 \sim \mathcal{N}(m_0, \Sigma_0), \quad \text{(Gaussian prior)}$$

Problem: Find conditional probability distribution of X_t given history of observation $\mathcal{Z}_t := \{Z_s; s \in [0,t]\}$

$$\mathsf{P}(X_t|\mathcal{Z}_t) = \ ?$$

J. Xiong, An introduction to stochastic filtering theory, 2008

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Kalman-Bucy filter

Kalman-Bucy filter: $P(X_t|\mathcal{Z}_t)$ is Gaussian $\mathcal{N}(m_t, \Sigma_t)$

Update for mean: $$\mathrm{d}m_t = (\text{linear dynamics}) + \underbrace{\mathsf{K}_t\,\mathrm{d}I_t}_{\text{correction}}$$

Update for covariance: $$\frac{\mathrm{d}\Sigma_t}{\mathrm{d}t} = \mathsf{Ric}(\Sigma_t) \quad \text{(Ricatti equation)}$$

Kalman gain: $$\mathsf{K}_t := \Sigma_t H^\top$$

Innovation process: $$\mathrm{d}I_t := \mathrm{d}Z_t - Hm_t\,\mathrm{d}t$$

Computational remark:

if state dimension is d $\Rightarrow$ covariance matrix is $d \times d$

$\Rightarrow$ computational complexity is $O(d^2)$

$\Rightarrow$ Not scalable for high-dim problems (e.g weather prediction)

R. E Kalman and R. S Bucy. New results in linear filtering and prediction theory, 1961

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Stochastic linear FPF and ensemble Kalman filter

Idea: Propagate particles $\{X_t^i\}_{i=1}^N \sim \mathsf{P}(X_t|\mathcal{Z}_t)$ instead of mean and covariance

$$\mathrm{d}X_t^i = (\text{linear dynamics}) + \underbrace{\mathsf{K}_t^{(N)}(\mathrm{d}I_t - \frac{1}{2}H(X_t^i - m_t^{(N)})\,\mathrm{d}t)}_{\text{correction}}, \quad X_0^i \overset{\text{i.i.d}}{\sim} p_0$$

where

$$\text{empirical mean:} \quad m_t^{(N)} := \frac{1}{N}\sum_{i=1}^N X_t^i$$

$$\text{empirical covariance:} \quad \Sigma_t^{(N)} := \frac{1}{N-1}\sum_{i=1}^N (X_t^i - m_t^{(N)})(X_t^i - m_t^{(N)})^\top$$

$$\text{empirical Kalman gain:} \quad \mathsf{K}_t^{(N)} := \Sigma_t^{(N)} H^\top$$

Exactness: If $N = \infty$ (mean-field limit), then $m_t^{(N)} = m_t$ and $\Sigma_t^{(N)} = \Sigma_t$

Computational remark: computational complexity is $O(Nd)$. Efficient when $d >> N$

Question: What is the approximation error when $N < \infty$?

G. Evensen. Sequential data assimilation with a nonlinear quasi-geostrophic model . . . 1994.
K. Bergemann and S. Reich. An ensemble Kalman-Bucy filter for continuous data assimilation, 2012
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Literature review

Background on ensemble Kalman filter and FPF

EnKf: [G. Evensen, 1994]

- Widely applied in geophysical sciences
- Exact only for linear Gaussian setting
- Two established forms of EnKF:
  (i) EnKF based on perturbed observation
  (ii) The square root EnKF

FPF: [T. Yang, et. al. 2012]

- Alternative to particle filter
- Does not suffer from particle degeneracy and admits lower simulation variance
- Exact for nonlinear non-Gaussian setting
- Generalization of the EnKF to non-linear setting
- Two forms of linear FPF:
  (i) Stochastic linear FPF (same as square-root EnKf)
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1) EnKf with perturbed observation

Assumption: System is stable and fully observable $(H^\top H = \rho I)$

- Convergence with $O(\frac{1}{\sqrt{N}})$ on finite time horizon: [Le Gland et. al. 2009]
- Convergence with $O(\frac{1}{\sqrt{N}})$ uniform in time [Del moral, et. al. 2016]

2) Deterministic FPF

Assumption: System is stabilizable and detectable

- Convergence with $O(\frac{e^{-\lambda t}}{\sqrt{N}})$ [Taghvaei and Mehta .(ACC) 2018]

3) Stochastic linear FPF or square root EnKF

- Error analysis: Subject of this work

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Finite-N system:

$$\mathrm{d}X_t^i = \text{(linear dynamics)} + \mathsf{K}_t^{(N)}\big(\,\mathrm{d}I_t - \frac{1}{2}H(X_t^i - m_t^{(N)})\,\mathrm{d}t\big), \quad X_0^i \overset{\text{i.i.d}}{\sim} p_0$$

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with empirical mean $m_t^{(N)}$ and covariance $\Sigma_t^{(N)}$

Mean-field limit:

$$\mathrm{d}\bar{X}_t = \text{(linear dynamics)} + \bar{\mathsf{K}}_t\big(\,\mathrm{d}\bar{I}_t - \frac{1}{2}H(X_t^i - \bar{m}_t)\,\mathrm{d}t\big), \quad \bar{X}_0 \sim p_0$$

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Error analysis:

1. Analysis of the mean-field system
2. Analysis of the converegnce of the finite-N system to the mean-field limit

$$\text{Finite-}N\text{ system} \overset{(2)}{\approx} \text{mean-field system} \overset{(1)}{=} \text{Kalman filter}$$

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Problem formulation

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Evolution of the mean and covariance

Finite-N system:

$$\mathrm{d}m_t^{(N)} = \underbrace{\text{(linear dynamics)} + \mathsf{K}_t^{(N)}\,\mathrm{d}I_t}_{\text{Kalman filter}} + \underbrace{\frac{\sigma_B}{\sqrt{N}}\,\mathrm{d}\tilde{B}_t}_{\text{stochastic term}}$$

$$\mathrm{d}\Sigma_t^{(N)} = \underbrace{\mathsf{Ric}(\Sigma_t^{(N)})\,\mathrm{d}t}_{\text{Kalman filter}} + \underbrace{\frac{\mathrm{d}M_t}{\sqrt{N}}}_{\text{stochastic term}}$$

Mean-field system system ($N=\infty$):

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Linear FPF is exact

Proposition

Consider the linear Gaussian filtering problem (X_t, Z_t), and the mean-field system $\bar{X}_t$.

1. If $\bar{m}_0 = m_0$ and $\bar{\Sigma}_0 = \Sigma_0$, then

$$\bar{m}_t = m_t, \quad \bar{\Sigma}_t = \Sigma_t$$

2. If the initial distribution is Gaussian $\bar{X}_0 \sim \mathcal{N}(m_0, \Sigma_0)$,

$$\bar{X}_t \sim \mathsf{P}(X_t | \mathcal{Z}_t)$$

Existence and uniqueness of the mean-field process

Mean-field system:

$$\mathrm{d}\bar{X}_t = (\text{linear dynamics}) + \bar{\mathsf{K}}_t(\,\mathrm{d}\bar{I}_t - \frac{1}{2}H(X_t^i - \bar{m}_t)\,\mathrm{d}t), \quad \bar{X}_0 \sim p_0$$
$$\bar{\mathsf{K}}_t = \bar{\Sigma}_t H^\top$$

- It is a McKean-Vlasov sde
- Fixed-point type technique is used to show existence of a unique mean-field process

Proposition (Existence and uniqueness)

The McKean-Vlasov sde has a unqiue strong solution on the space $C([0,T],\mathbb{R}^d)$ such that $\mathsf{E}[\sup_t |\bar{X}_t|^2] < \infty$

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Stability of the mean-field process

Define $\bar{\xi}_t = \bar{X}_t - \bar{m}_t$, then

$$\mathrm{d}\bar{m}_t = \text{(Kalman filter)}$$
$$\mathrm{d}\bar{\xi}_t = (A - \frac{1}{2}\bar{\mathsf{K}}_t H)\bar{\xi}_t + \sigma_B \, \mathrm{d}\bar{B}_t$$

Assumptions:

- The system (A, H) is detectable and (A, σ_B) is stabilizable (for stability of $\bar{m}_t$)
- The covariance matrix $\Sigma_B = \sigma_B \sigma_B^\top \succ 0$ (for stability of $\bar{\xi}$)

Proposition

- Let $\bar{X}_t \sim \pi_t$ and $\tilde{X}_t \sim \tilde{\pi}_t$ be solutions to the mean-field system with different initial condition. Then

$$W_2(\pi_t, \tilde{\pi}_t) \leq M e^{-\beta t}$$

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Proposition (convergence)

Assume $d = 1$ (scalar case). Then

$$\mathsf{E}[|\Sigma_t^{(N)} - \Sigma_t|^{2p}]^{1/p} \leq (\text{const.})\,\frac{e^{-\beta t}}{N} + \frac{(\text{const.})}{N}$$

Assume the matrix A is stable. Then

$$\mathsf{E}[|m_t^{(N)} - m_t|^2] \leq (\text{const.})\,\frac{e^{-2\mu(A)t}}{N} + \frac{(\text{const.})}{N}$$

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Proposition

Consider the stochastic linear FPF for the linear Gaussian problem where the system is stable. Then

$$\mathsf{E}[\big|\frac{1}{N}\sum_{i=1}^{N} f(X_t^i) - \mathsf{E}[f(X_t)|\mathcal{Z}_t]\big|^2] \leq \frac{\text{(const)}}{N}, \quad \forall f \in C_b(\mathbb{R}^d)$$

$\boxed{\text{The empirical distribution converges to the posterior disttribution}}$

A. Sznitman. Topics in propagation of chaos, 1991

Conclusions and future work

- We proved the convergence and provided error-bounds for a stable and scalar system
- Recent work [Bishop and Del Moral, 2018] proved error-bounds under the assumption that the system is fully observable (H is full-rank)
- Is it possible to do the error analysis under stabilizable and detectable assumption? open problem
- The are many finite-N system that have the same mean-field limt. Should we change the finie-N system? Can dual formulation be helpful?

Thank you for your attention!

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