Optimality vs Stability Trade-off in Ensemble Kalman Filters

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Joint work with Prashant Mehta[*] and Tryphon Georgiou[+]

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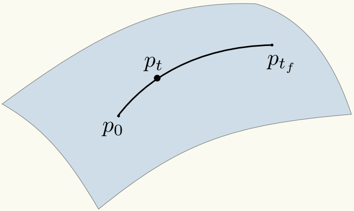
Sep 15, 2022

Realizing probability flows
with stochastic processes



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$$\text{Law}(X_t) = p_t, \quad \forall t \geq 0.$$

Significance:

- interacting particle sampling algorithms (e.g. SVGD)
- simulating Wasserstein gradient flows
- diffusion-based generative models
- controlled particle filtering algorithms $\rightarrow$ this talk

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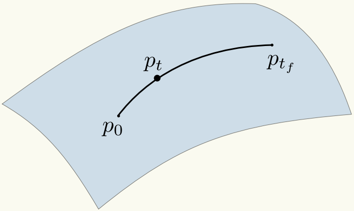

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Realizing probability flows
Example

- Assume p_t is a one-dimensional Gaussian $N(0, \Sigma_t)$ where $\Sigma_t = 1 + 2t$.
- Assume X_t is of the form

$$\mathrm{d}X_t = u_t \mathrm{d}t + v_t \mathrm{d}B_t, \quad X_0 \sim p_0.$$

- Design u_t and v_t such that $\mathsf{Law}(X_t) = p_t$ for all $t \geq 0$.
- Solution:

$$\mathrm{d}X_t = \sqrt{2}\mathrm{d}B_t, \quad X_0 \sim N(0,1)$$

- Why?

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Question: Are there any other realizations?

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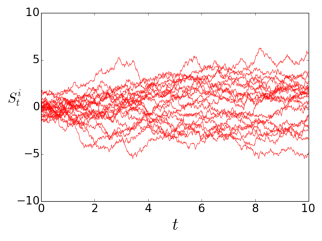

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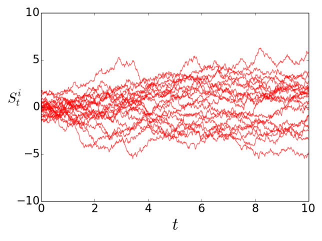

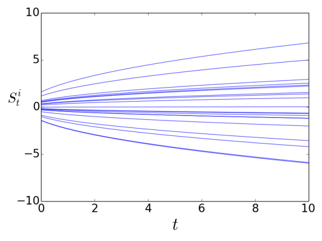

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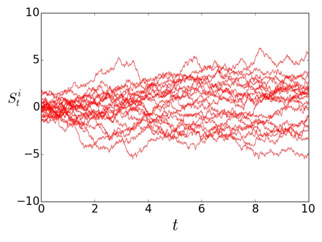
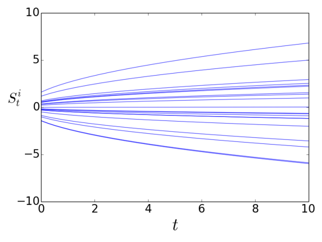
- In the mean-field limit, do they exhibit different (stability) properties?
- In finite-N setting, do they exhibit different approximation error?





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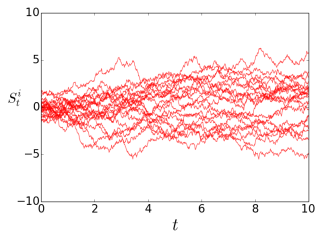

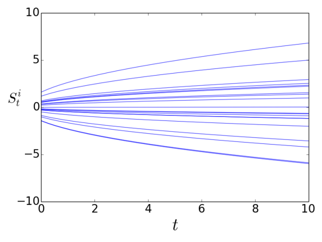

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Filtering problem

Model:

$$\textbf{state process}: \quad \{X_t\}_{t\geq 0}$$

$$\textbf{observation process}: \quad \{Z_t\}_{t\geq 0}$$

$$\textbf{posterior dist.}: \quad \pi_t := \mathbb{P}(X_t|\mathcal{Z}_t) \quad \text{where} \quad \mathcal{Z}_t := \sigma(Z_s; s \in [0,t])$$

Objective: compute the posterior distribution π_t.

Approach:

- realize $\{\pi_t\}_{t\geq 0}$ with a stochastic process $\{\bar{X}_t\}_{t\geq 0}$ s.t.

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- simulate $\bar{X}_t$ as a system of N interacting particles $\{\bar{X}_t^1, \ldots, \bar{X}_t^N\}$,

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state process : $\mathrm{d}X_t = AX_t\mathrm{d}t + \sigma_B \mathrm{d}B_t$

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Kalman filter: posterior dist. π_t is Gaussian $N(m_t, \Sigma_t)$ where

$$\mathrm{d}m_t = Am_t + \mathsf{K}_t(\mathrm{d}Z_t - Hm_t\mathrm{d}t) =: \mathcal{T}_t(m_t, \Sigma_t)$$
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- Realize π_t with a stochastic process $\bar{X}_t$.
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Ensemble Kalman filter
Three established forms

- EnKF with perturbed observation [Reich 2011]

$$\textbf{P-EnKF:}\quad \mathrm{d}\bar{X}_t = A\bar{X}_t\mathrm{d}t + \sigma_B \mathrm{d}\bar{B}_t + \bar{\mathsf{K}}_t(\mathrm{d}Z_t - H\bar{X}_t\mathrm{d}t - \mathrm{d}\bar{W}_t)$$

- Square-root EnKF [Bergemann & Reich 2012]

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- Deterministic EnKF [Taghvaei & Mehta 2016]

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$$\text{Law}(\bar{X}_t) = \pi_t, \quad \forall t \geq 0, \quad \text{if} \quad \underline{\text{Law}(\bar{X}_0) = \pi_0}.$$

- What if $\text{Law}(\bar{X}_0) \neq \pi_0$? Does $\text{Law}(\bar{X}_t) \to \pi_t$ as $t \to \infty$?

Stability

Assume the pair (A, σ_B) is controllable and the pair (A, H) is observable. Then,

$$\mathbb{E}[\mathcal{W}_2(\bar{\pi}_t, \pi_t)] \leq Ce^{-\lambda_0 t}\mathcal{W}_2(N(\bar{m}_0, \bar{\Sigma}_0), N(m_0, \Sigma_0)) + e^{\int_0^t G_s \mathrm{d}s}\mathcal{W}_2(N(\bar{m}_0, \bar{\Sigma}_0), \bar{\pi}_0)$$

Ensemble Kalman filter
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$$\text{P-EnKF:}\quad \mathbb{E}[\mathcal{W}_2(\bar{\pi}_t, \pi_t)] \leq (\text{const.})e^{-\lambda_0 t}$$

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$$\text{D-EnKF:}\quad \mathbb{E}[\mathcal{W}_2(\bar{\pi}_t, \pi_t)] \geq (\text{const.})$$

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- system of interacting particles:

$$\mathrm{d}X_t^i = \mathcal{T}_t(m_t^{(N)}, \Sigma_t^{(N)}) + G_t(X_t^i - m_t^{(N)})\mathrm{d}t + r_t \mathrm{d}B_t^i + q_t \mathrm{d}W_t^i$$

- objective: analyze the empirical approximation error

$$\bar{\pi}_t \approx \pi_t^{(N)} := \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^i}$$

Error analysis for empirical variance

Assume $\sup_{t\geq 0} \mathbb{E}[(r_t^2 + q_t^2)\bar{\Sigma}_t] = M < \infty$. Then, the error in empirical variance

$$\lim_{t\to\infty} \mathbb{E}[(\Sigma_t^{(N)} - \Sigma_t)^2] \leq (\text{const.})\frac{M}{N}$$

- Error is due to stochastic terms in the dynamics

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Summary and conclusions

- General form of EnKF:

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$$\text{s.t.} \quad 2G_t\bar{\Sigma}_t + r_t^2 + q - t^2 = \mathrm{Ricc}(\bar{\Sigma}_t)$$

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Algorithm	G_t	r_t	q_t	Stability rate	finite-N error
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- a trade-off between stability and finite-N error
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