Bias-Variance Tradeoff in Numerical Solution to the Poisson Equation

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Jan 13, 2017

ILLINOIS

Poisson equation: $$-\frac{1}{\rho(x)}\nabla\cdot(\rho(x)\nabla\phi(x)) = h(x) - \hat{h}$$
$$\int_{\mathbb{R}^d} \phi(x)\rho(x)\,\mathrm{d}x = 0$$

- $\rho : \mathbb{R}^d \to \mathbb{R}^+$ (prob. density)
- $h : \mathbb{R}^d \to \mathbb{R}$ (given function), $\hat{h} := \int h(x)\rho(x)\,\mathrm{d}x$
- $\phi : \mathbb{R}^d \to \mathbb{R}$ (solution)

Problem:

Given: $\{X^1, \ldots, X^N\} \overset{\text{i.i.d}}{\sim} \rho$

Find: $\{\nabla\phi(X^1), \ldots, \nabla\phi(X^N)\}$ (approximately)

Almost like a statistical learning problem

R. S. Laugesen, P. G. Mehta, S. P. Meyn, and M. Raginsky. Poisson Equation in Nonlinear Filtering. SICON, 2015

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Feedback Particle Filter

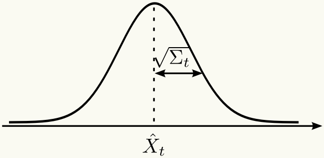
Generalization of the Kalman Filter

Kalman Filter:

$$\mathrm{d}X_t = AX_t\,\mathrm{d}t + \mathrm{d}B_t$$
$$\mathrm{d}Z_t = HX_t\,\mathrm{d}t + \mathrm{d}W_t$$

$\mathsf{P}(X_t|\mathcal{Z}_t) =$ Gaussian $N(\hat{X}_t, \Sigma_t)$,

$$\mathrm{d}\hat{X}_t = A\hat{X}_t\,\mathrm{d}t + \mathsf{K}_t(\,\mathrm{d}Z_t - H\hat{X}_t\,\mathrm{d}t)$$
$$\frac{\mathrm{d}\Sigma_t}{\mathrm{d}t} = \ldots \text{(Riccati equation)}$$



Feedback Particle Filter:

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$\mathsf{P}(X_t|\mathcal{Z}_t) \approx$ empirical dist. $\{X^1, \ldots, X^N\}$,

$$\mathrm{d}X_t^i = a(X_t^i)\,\mathrm{d}t + \mathrm{d}B_t^i + \mathsf{K}_t(X_t^i) \circ (\,\mathrm{d}Z_t - \frac{h(X_t^i) + \hat{h}_t}{2}\,\mathrm{d}t)$$

Challenge: Compute the gain function $\mathsf{K}_t := \nabla\phi$ from Poisson eq.

T. Yang, P. G. Mehta, and S. P. Meyn. feedback particle filter, *TAC*, 2013
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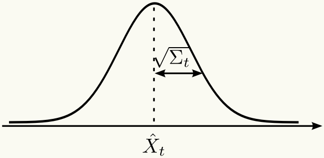

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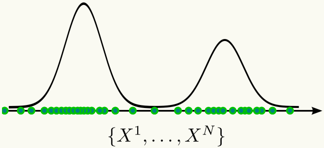

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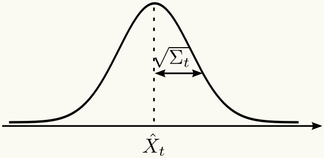

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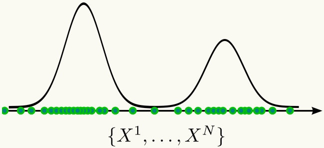

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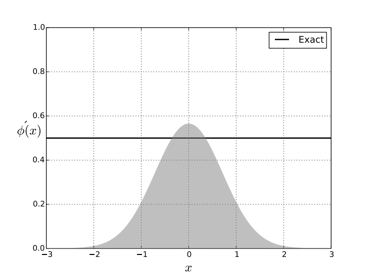
Gaussian distribution linear h

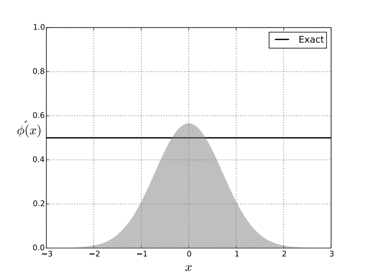


$\nabla\phi(x) = $ constant (Kalman gain)

Bimodal distribution linear h

$\nabla\phi(x) = \ldots$ (Nonlinear gain)

Gaussian distribution
linear h



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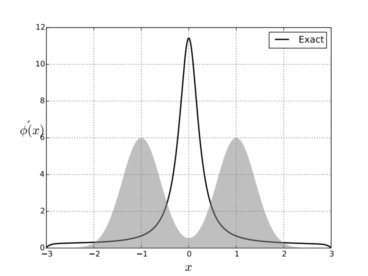

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Literature Review
Poisson equation and weighted Laplacian

Poisson equation: $$-\frac{1}{\rho}\nabla\cdot(\rho\nabla\phi) = h - \hat{h}$$

Weighted Laplacian: $$\Delta_\rho\phi := \frac{1}{\rho}\nabla\cdot(\rho\nabla\phi) = \Delta\phi + \nabla\log\rho\cdot\nabla\phi$$

PDE

- Markov Diffusion operators [D. Bakry, et. al. 2013]
- Heat kernels [A. Grigoryan, 2009]

Stochastic analysis

- Simulation and optimization theory for Markov models [S. Meyn, R. Tweedie, 2012]

Statistical learning

- Nonlinear dimensionality reduction [M. Belkin, 2003]
- Diffusion maps [R. Coifman, S. Lafon, 2006]
- Spectral clustering [M. Hein, et. al. 2006]

Three Formulations of the Poisson Equation

P) Weak formulation: (Galerkin)

$$\langle \nabla\phi, \nabla\psi \rangle = \langle h - \hat{h}, \psi \rangle, \quad \forall \psi \in H^1(\mathbb{R}^d, \rho)$$

where $\langle f, g \rangle := \int f(x)g(x)\rho(x)\,\mathrm{d}x$

S) Semigroup formulation: (kernel-based)

$$\phi = P\phi + \tilde{h}$$

where $P := e^{\epsilon \Delta_\rho}$ and $\tilde{h} := \int_0^t e^{s\Delta_\rho}(h - \hat{h})\,\mathrm{d}s$

3) Variational formulation: (Neural net ?)

$$\min_{\phi \in H_0^1(\mathbb{R}^d, \rho)} \mathsf{E}\left[\frac{1}{2}|\nabla\phi(X)|^2 - \phi(X)(h(X) - \hat{h})\right]$$

where $X \sim \rho$

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Strong form:

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Weak form:

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Galerkin approximation:

$$\langle\nabla\phi^{(M)},\nabla\psi\rangle = \langle h-\hat{h},\psi\rangle, \quad \forall\psi\in S$$

where $S = \text{span}\{\psi_1,\ldots,\psi_M\}$

Empirical approximation

$$\frac{1}{N}\sum_{i=1}^{N}\nabla\phi^{(M)}(X^i)\cdot\nabla\psi(X^i) = \frac{1}{N}\sum_{i=1}^{N}(h(X^i)-\hat{h})\psi(X^i), \quad \forall\psi\in S$$

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where $X^i\sim\rho$

- Select basis functions $\{\psi_1, \dots, \psi_M\}$
- Express the approximate solution as

$$\phi^{(M,N)}(x) = \sum_{m=1}^{M} c_m \psi_m(x)$$

- Obtain $c = (c_1, \dots, c_M)$ by solving

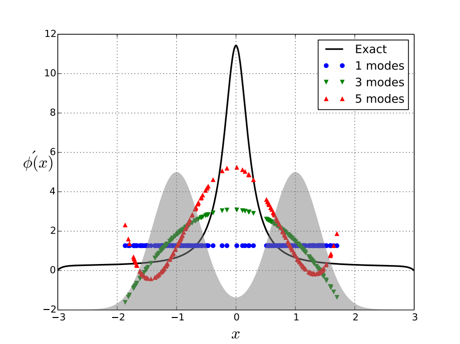
$$Ac = b$$

where

$$A_{ml} = \langle \nabla\psi_m, \nabla\psi_l \rangle \approx \frac{1}{N}\sum_{i=1}^{N} \nabla\psi_m(X^i) \cdot \nabla\psi_l(X^i)$$

$$b_m = \langle \psi_m, h \rangle \approx \frac{1}{N}\sum_{i=1}^{N} \psi_m(X^i) h(X^i) - \hat{h})$$

Galerkin Approximation
Numerical result



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- Singularity of A
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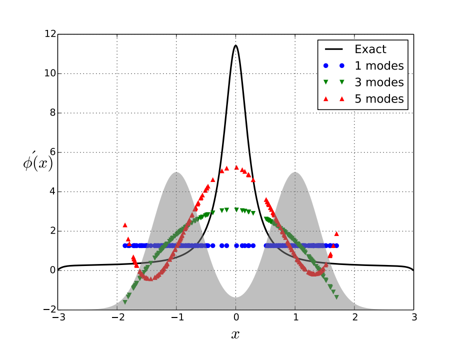

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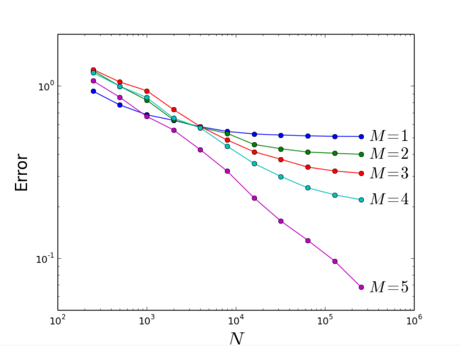
Special case: The basis functions are eigenfunctions of Δ_ρ

$$\underbrace{\mathsf{E}\left[\|\nabla\phi - \nabla\phi^{(M,N)}\|_{L^2}\right]}_{\text{Total error}} \leq \underbrace{\frac{1}{\sqrt{\lambda_M}}\|h - \Pi_S h\|_{L^2}}_{\text{Bias}} + \underbrace{\frac{1}{\sqrt{N}}\|h\|_\infty \sqrt{\sum_{m=1}^{M} \frac{1}{\lambda_m}}}_{\text{Variance}}$$



Poisson equation: $-\Delta_\rho \phi = h - \hat{h}$

Semigroup identity: $e^{\epsilon \Delta_\rho} = I + \int_0^\epsilon e^{s\Delta_\rho} \Delta_\rho \, \mathrm{d}s$

Semigroup formulation:

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where $\tilde{h} := \int_0^\epsilon e^{s\Delta_\rho} (h - \hat{h}) \, \mathrm{d}s$

Kernel representation:

$$\phi(x) = \int \tilde{k}_\epsilon(x,y) \phi(y) \rho(y) \, \mathrm{d}y + \tilde{h}(x)$$

Empirical approximation:

$$\phi(x) = \frac{1}{N} \sum_{i=1}^N \tilde{k}_\epsilon(x, X^i) \phi(X^i) + \tilde{h}(x)$$

But $\tilde{k}_\epsilon(x,y) = ?$

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where $\tilde{h} := \int_0^\epsilon e^{s\Delta_\rho} (h - \hat{h}) \, \mathrm{d}s$

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$$\phi(x) = \int \tilde{k}_\epsilon(x,y) \phi(y) \rho(y) \, \mathrm{d}y + \tilde{h}(x)$$

Empirical approximation:

$$\phi(x) = \frac{1}{N} \sum_{i=1}^N \tilde{k}_\epsilon(x, X^i) \phi(X^i) + \tilde{h}(x)$$

But $\tilde{k}_\epsilon(x,y) = ?$

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Kernel-based Approximation

Special case: $\rho = 1$

$$e^{\epsilon\Delta} f(x) = \int g_\epsilon(x,y) f(y)\,\mathrm{d}y. \quad \text{(for all } \epsilon > 0\text{)}$$

where g_ϵ is the Gaussian kernel.

In general:

$$e^{\epsilon\Delta_\rho} f(x) \approx \int \frac{1}{n_\epsilon(x)} \frac{g_\epsilon(x,y)}{\sqrt{\int g_\epsilon(y,z)\rho(z)\,\mathrm{d}z}} f(y)\rho(y)\,\mathrm{d}y := T_\epsilon f(x) \quad \text{(for } \epsilon \downarrow 0\text{)}$$

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R. Coifman, S. Lafon, Diffusion maps, *Applied and computational harmonic analysis*, 2006,
M. Hein, J. Audibert, U. Von Luxburg, Convergence of graph Laplacians on random neighborhood graphs, *JLMR*, 2007

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Gradeint:

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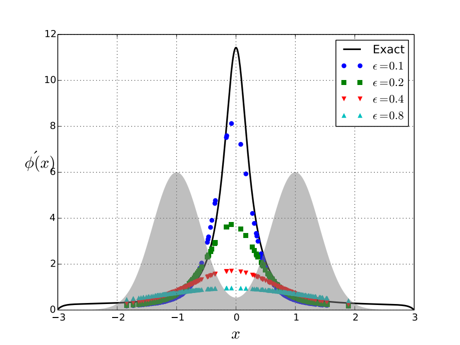
Kernel-based approximation
Numerical result



Properties

1. No singularity
2. Easy extension to Manifolds [C. Zhang, et. al. CDC 2015]
3. Better error bounds
4. Computational cost $O(N^2)$ (good in high dimensions)

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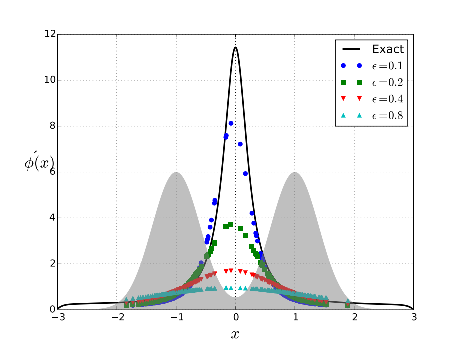

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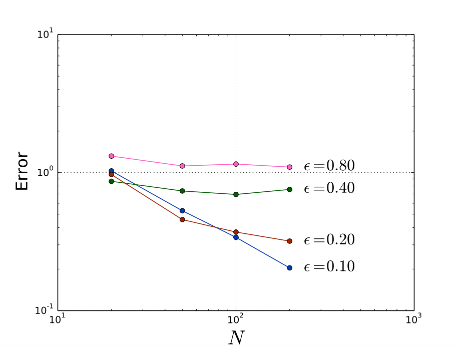
$$\underbrace{\mathsf{E}\left[\|\nabla\phi - \nabla\phi_\epsilon^{(N)}\|_2\right]}_{\text{Total error}} \le \underbrace{O(\epsilon)}_{\text{Bias}} + \underbrace{O(\frac{1}{\epsilon^{1+d/4}\sqrt{N}})}_{\text{Variance}}$$



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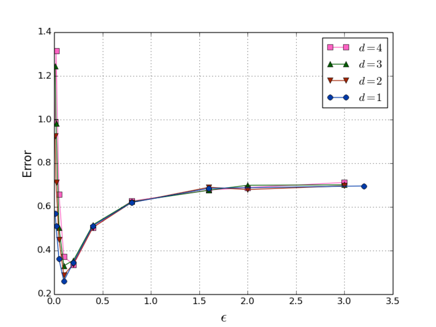
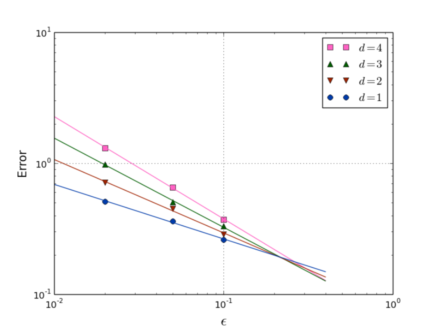
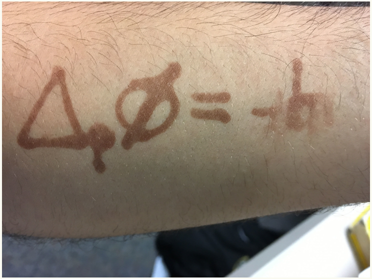




Final slide

Thank you for your attention!



Poisson equation, almost everywhere