Linear Feedback Particle Filter

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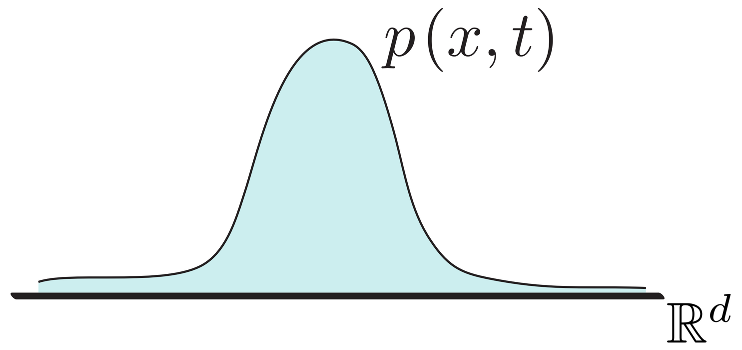
Nonlinear Filtering Problem

Signal model: $\mathrm{d}X_t = a(X_t)\,\mathrm{d}t + \mathrm{d}B_t, \quad X_0 \sim p_0(\cdot)$

Observation model: $\mathrm{d}Z_t = h(X_t)\,\mathrm{d}t + \mathrm{d}W_t$

Problem: What is X_t given $\mathcal{Z}_t := \sigma(Z_s : 0 \le s \le t)$

Answer in terms of posterior: $\mathsf{P}(X_t|\mathcal{Z}_t)$



Posterior is solution to Kushner-Stratonovich PDE (hard to solve!)

Feedback Particle Filter

A system of N particles,

$$\mathrm{d}\tilde{X}_t^i = a(\tilde{X}_t^i)\,\mathrm{d}t + \mathrm{d}\tilde{B}_t^i + K(\tilde{X}_t^i, t)\left[\mathrm{d}Z_t - \frac{h(\tilde{X}_t^i) + \hat{h}_t}{2}\mathrm{d}t\right], \quad \text{for } i = 1, \ldots, N$$

Posterior is approximated with particles,

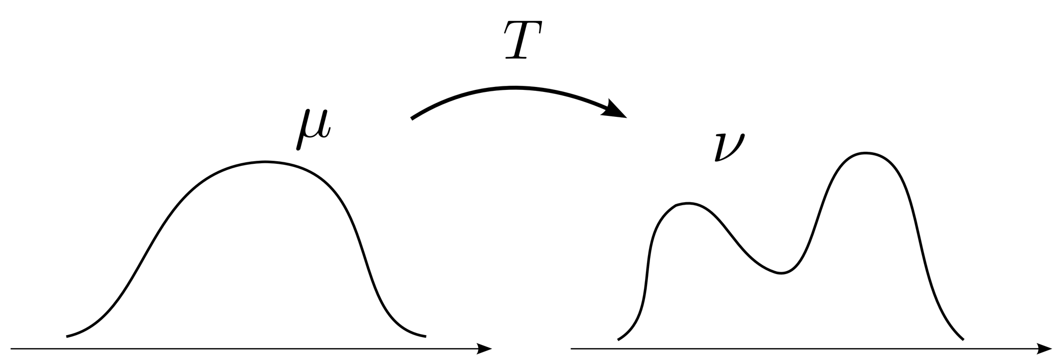
$$\mathsf{P}(X_t \in A|\mathcal{Z}_t) \approx \frac{1}{N}\sum_{i=1}^{N} \delta[X_t^i \in A], \quad \forall A \in \mathcal{B}(\mathbb{R}^d)$$

T. Yang, et al. TAC, (2013)

Objective of This Work

1. Reduce simulation variance
2. Connect to optimal transport
3. Examine optimality of FPF

Optimal Transportation



Let μ and ν be measures on $\mathbb{R}^d$, and T a map from $\mathbb{R}^d$ to $\mathbb{R}^d$,

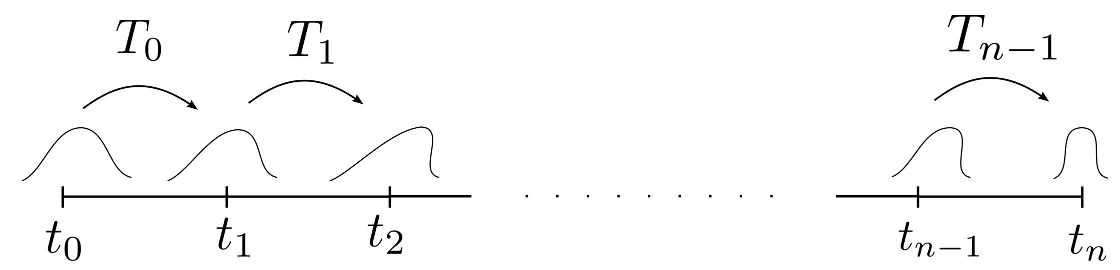
- $T_\# \mu = \nu$ denotes the push-forward of μ to ν, i.e,
$$\nu(A) = \mu(T^{-1}(A)), \quad \forall A \in \mathcal{B}(\mathbb{R}^d)$$
- Wasserstein distance between μ and ν is,
$$W_2^2(\mu,\nu) = \min_T \int_{\mathbb{R}^d} |T(x) - x|^2 \,\mathrm{d}\mu(x), \quad \text{s.t} \quad T_\#\mu = \nu$$
- The optimal transport map is the minimizer T^*.

Approach

1. Design an optimal control law U_t,
$$\mathrm{d}\tilde{X}_t = \mathrm{d}U_t(\tilde{X}_t)$$
such that
$$\mathsf{P}(\tilde{X}_t|\mathcal{Z}_t) = \mathsf{P}(X_t|\mathcal{Z}_t)$$
2. Simulate N samples of $\tilde{X}_t$,
$$\mathrm{d}\tilde{X}_t^i = \mathrm{d}U_t(X_t^i), \quad \text{for } i = 1, \ldots, N$$

Design am Optimal Control Law

1. Consider sampling instances $\{t_0, t_1, \ldots, t_n\}$ with $t_{k+1} - t_k = \Delta t$, and denote $\mathsf{P}_k = \mathsf{P}_{X_{t_k}|\mathcal{Z}_{t_k}}$,



2. Initialize $\tilde{X}_t$ according to true initial distribution,
$$\tilde{X}_0 \sim \mathsf{P}_0$$
3. Find optimal transport map for each time step, $t_k \to t_{k+1}$,
$$T_k = \arg\min_T \int_{\mathbb{R}^n} |T(x) - x|^2 \,\mathrm{d}\mathsf{P}_k(x), \quad \text{s.t} \quad T_\#\mathsf{P}_k = \mathsf{P}_{k+1}$$
4. Evolve $\tilde{X}_t$ according to the optimal map,
$$\tilde{X}_{t_{k+1}} = T_k(\tilde{X}_{t_k})$$
5. Take the limit $\Delta t \to 0$
$$\mathrm{d}\tilde{X}_t = \mathrm{d}\tilde{U}_t(\tilde{X}_t)$$

Example: Linear Gaussian Filtering

Consider the linear filtering problem,

$$\mathrm{d}X_t = AX_t\,\mathrm{d}t + \mathrm{d}B_t, \quad X_0 \sim N(\mu_0, \Sigma_0)$$
$$\mathrm{d}Z_t = CX_t\,\mathrm{d}t + \mathrm{d}W_t$$

where $X_t \in \mathbb{R}^d$, $Z_t \in \mathbb{R}^m$, $\{B_t\}$ and $\{W_t\}$ are independent Brownian motions, with Identity covariance matrix, then the optimal control law for this system is

$$\mathrm{d}\tilde{X}_t = A\tilde{\mu}_t\,\mathrm{d}t + \tilde{K}_t(\mathrm{d}Z_t - C\tilde{\mu}_t\,\mathrm{d}t) + G_t(\tilde{X}_t - \tilde{\mu}_t)\,\mathrm{d}t, \quad \tilde{X}_0 \sim N(\mu_0, \Sigma_0)$$

where

$$\tilde{K}_t = \tilde{\Sigma}_t C^T, \quad \tilde{\mu}_t = \mathsf{E}[\tilde{X}_t], \quad \tilde{\Sigma}_t = \mathsf{Cov}[\tilde{X}_t]$$

and G_t is the unique solution to the following equation,

$$G_t\tilde{\Sigma}_t + \tilde{\Sigma}_t G_t = A\tilde{\Sigma}_t + \tilde{\Sigma}_t A^T + I - \tilde{\Sigma}_t C^T C \tilde{\Sigma}_t$$

Scalar case: In the case $X_t, Z_t \in \mathbb{R}$, we have

$$\mathrm{d}\tilde{X}_t = A\tilde{X}_t\,\mathrm{d}t + +\frac{1}{2\tilde{\Sigma}_t}(\tilde{X}_t - \tilde{\mu}_t)\,\mathrm{d}t + \tilde{K}_t(\mathrm{d}Z_t - \frac{C\tilde{X}_t + C\tilde{\mu}_t}{2}\mathrm{d}t), \quad \tilde{X}_0 \sim N(\mu_0, \Sigma_0)$$

Numerical Example

Consider the diffusion,

$$\mathrm{d}X_t = \mathrm{d}B_t, \quad X_0 \sim N(0,1)$$

where $\{B_t\}$ is the standard Brownian motion. In this example we compare Monte-Carlo with optimal transportation to approximate P_{X_t}.

Monte-Carlo

$$\mathrm{d}X_t^i = \mathrm{d}W_t^i, \quad X_0^i \overset{i.i.d}{\sim} N(0,1)$$

for $i = 1, \ldots, N$ where

$\{W_t^1\}, \ldots, \{W_t^n\}$ are ind. W.P

Optimal transport

$$\mathrm{d}X_t^i = \frac{1}{2\Sigma_t^{(N)}}(X_t^i - \hat{\mu}_t^{(N)}), \quad X_0^i \overset{i.i.d}{\sim} N(0,1)$$

for $i = 1, \ldots, N$ where

$$\mu_t^{(N)} = \frac{1}{N}\sum_{i=1}^{N} X_t^i, \quad \Sigma_t^{(N)} = \frac{1}{N}\sum_{i=1}^{N}(X_t^i - \mu_t^{(N)})^2$$

Approximation:

$$\mathsf{E}[f(X_t)] \approx \frac{1}{N}\sum_{i=1}^{N} f(X_t^i)$$

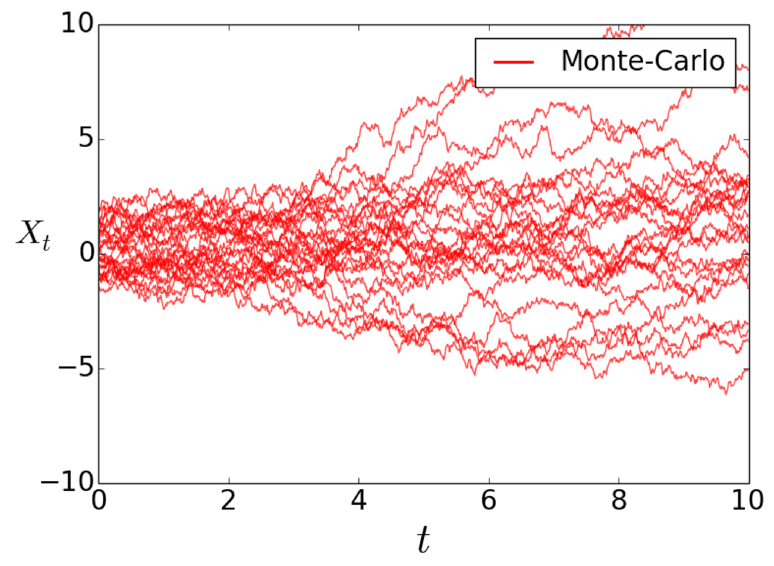
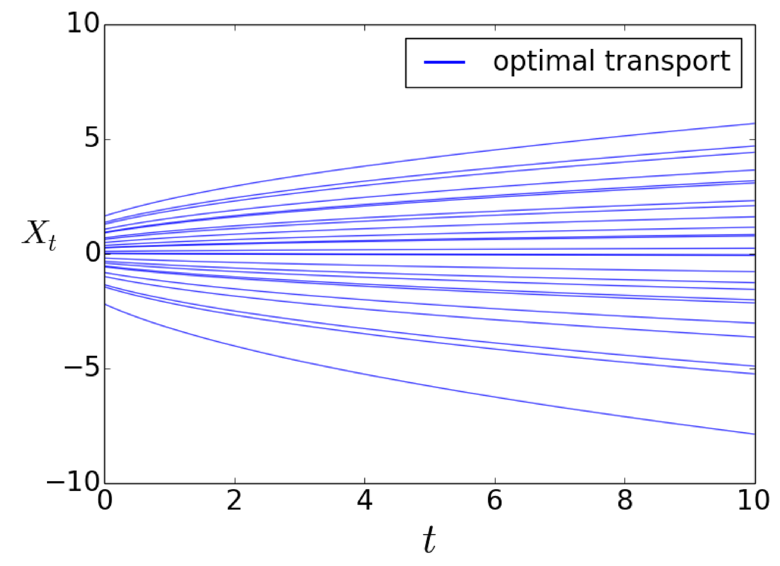


Figure : Trajectory of particles



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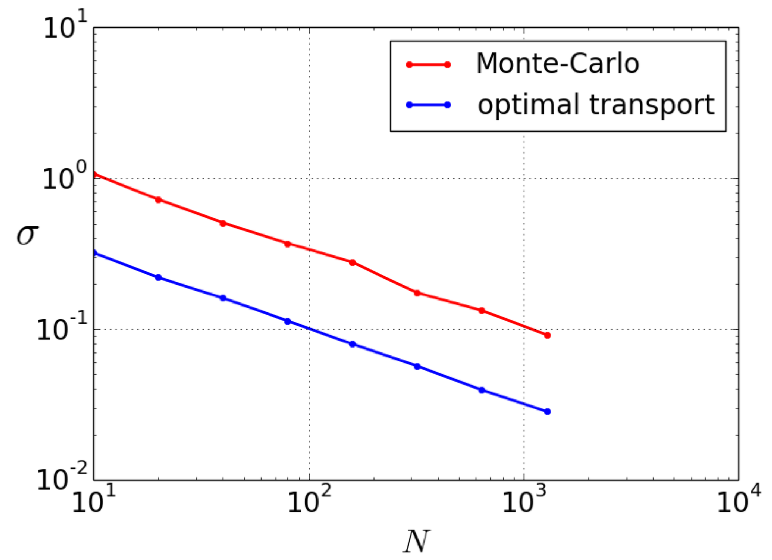

Figure : Simulation variance for $\mathsf{E}[X_t]$

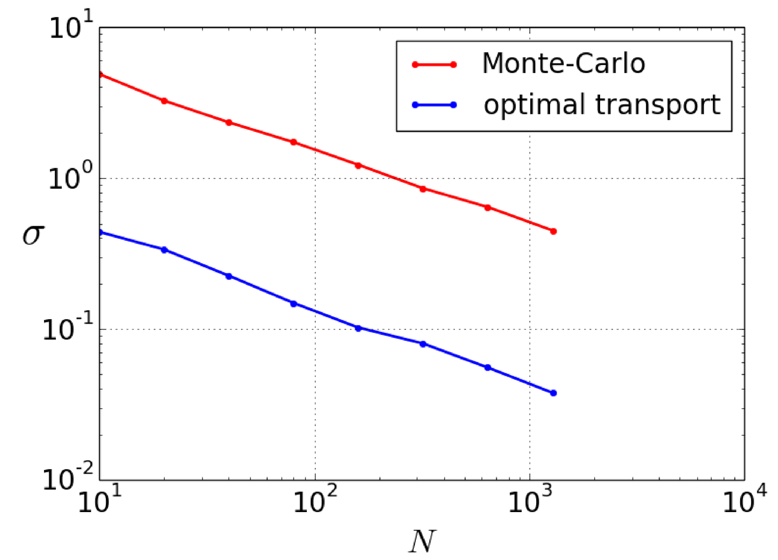

Figure : Simulation variance for $\mathsf{E}[X_t^2]$

Future Work

- Extend to nonlinear case
- Find an optimization formulation

Aknowledgement

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