Mean Field Optimal Control Formulation for Global Optimization

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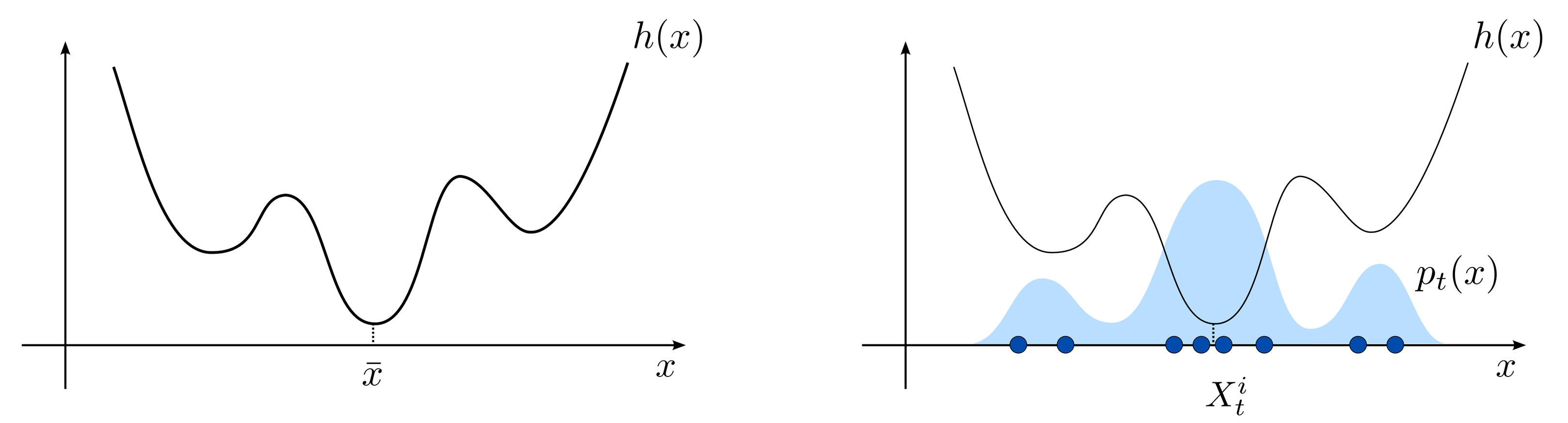
Global Optimization

Problem:

$$\min_{x \in \mathbb{R}^d} h(x)$$

- $h : \mathbb{R}^d \to \mathbb{R}$
- $\bar{x} := \arg\min_x h(x)$



Controlled Particle Filter Approach

Bayesian model:

$$p_t^*(x) = \frac{p_0^*(x)e^{-th(x)}}{\int p_0^*(y)e^{-th(y)}\,\mathrm{d}y} \to \delta_{\bar{x}}, \quad \text{as} \quad t \to \infty$$

Standard algorithm: Importance sampling and resampling [Wang, et. al. WSC (2010)]

Control-based algorithm: (Inspired by Feedback Particle Filter [Yang, et. al. TAC (2013)]

$$\text{ODE:} \quad \frac{\mathrm{d}X_t^i}{\mathrm{d}t} = -\nabla\phi_t(X_t^i), \quad X_0^i \sim p_0^*$$

$$\text{PDE:} \quad -\frac{1}{p_t(x)}\nabla\cdot(p_t(x)\nabla\phi_t(x)) = h(x) - \hat{h}_t$$

- $\{X_t^i\}_{i=1}^N \in \mathbb{R}^d$ is the state of the i^{th} particle
- p_t is the density of X_t^i
- $\hat{h}_t := \int h(x)p_t(x)\,\mathrm{d}x$

Consistency: The density p_t of X_t^i is equal to the Bayesian density

$$p_t = p_t^*$$

Objective of this work

Derive an optimal control formulation for the algorithm

Related work: Brocket, AMS/IP Stud. Adv. Math. (2007), Huang, et. al. TAC (2007) , Bensoussan et. al. IJPAM (2015), Chen, et. al. TAC (2016)

Optimal control formulation

Problem:

$$\text{Minimize:} \quad J(u) := \int_0^T \underbrace{L(\rho_t, u_t)}_{\text{Lagrangian}}\,\mathrm{d}t + \underbrace{\int h(x)\rho_T(x)\,\mathrm{d}x}_{\text{terminal cost}}$$

$$\text{Constraint:} \quad \frac{\partial \rho_t}{\partial t} + \nabla\cdot(\rho_t u_t) = 0, \quad \rho_0(x) = p_0^*(x)$$

where

$$L(\rho, u) = \int_{\mathbb{R}^d}\left[\frac{1}{2}\left|\frac{1}{\rho(x)}\nabla\cdot(\rho(x)u(x))\right|^2 + \frac{1}{2}|h(x) - \hat{h}|^2\right]\rho(x)\,\mathrm{d}x$$

Dynamic Programming

Define

Value function: $V(\rho, t) := \inf_u \left[\int_t^T L(\rho_s, u_s)\,\mathrm{d}s + \int h(x)\rho_T(x)\,\mathrm{d}x\right]$

Hamiltonian: $\mathrm{H}(\rho, q, u) := \mathrm{L}(\rho, \mathrm{u}) - \int_{\mathbb{R}^d} \mathrm{q}(\mathrm{x})\nabla\cdot(\rho(\mathrm{x})\mathrm{u}(\mathrm{x}))\,\mathrm{dx}$

then

$$\frac{\partial V}{\partial t}(\rho, t) + \inf_u \mathrm{H}\left(\rho, \frac{\partial V}{\partial \rho}(\rho, t), u\right) = 0,$$

$$V(\rho, T) = \int h(x)\rho(x)\,\mathrm{d}x$$

Solution to the DP equation

- The value function is

$$V(\rho, t) = \int h(x)\rho(x)\,\mathrm{d}x$$

$$\frac{\partial V}{\partial \rho}(\rho, t) = h$$

- The optimal control solves the pde

$$\frac{1}{\rho(x)}\nabla\cdot(\rho(x)u(x)) = h(x) - \hat{h}, \quad \forall x \in \mathbb{R}^d$$

Hamilton's equations

$$\boxed{\begin{aligned} \frac{\mathrm{d}X_t^i}{\mathrm{d}t} &= u_t(X_t^i), \quad X_0^i \sim p_0^* \\ u_t &= \arg\min_{\nu} \mathrm{H}(\rho_t, h, \nu) \end{aligned}}$$

Quadratic Gaussian case

$$p_0^* \quad \text{is} \quad \text{Gaussian} \ N(m_0, \Sigma_0)$$

$$h(x) = \frac{1}{2}(x - \bar{x})^T H(x - \bar{x})$$

then

$$\rho_t \quad \text{is} \quad \text{Gaussian} \ N(m_t, \Sigma_t)$$

$$u(x) = \underbrace{-\mathrm{K_t}(\mathrm{x} - \mathrm{m_t}) - \mathrm{b_t}}_{\text{Affine control law}}$$

where

$$b_t = \int_{\mathbb{R}^d} x(h(x) - \hat{h}_t)\rho_t(x)\,\mathrm{d}x$$

$$\mathrm{K_t}\Sigma_t + \Sigma_t\mathrm{K_t} = \int_{\mathbb{R}^d}(x - m)(x - m)^T(h(x) - \hat{h}_t)\rho_t(x)\,\mathrm{d}x$$

Affine approximation of the control law

Particle filter:

$$\frac{\mathrm{d}X_t^i}{\mathrm{d}t} = -\mathrm{K}_t^{(\mathrm{N})}(\mathrm{X}_t^i - \mathrm{m}_t^{(\mathrm{N})}) - \mathrm{b}_t^{(\mathrm{N})}$$

where $(m_t^{(N)}, \Sigma_t^{(N)})$ is the empirical mean and variance and

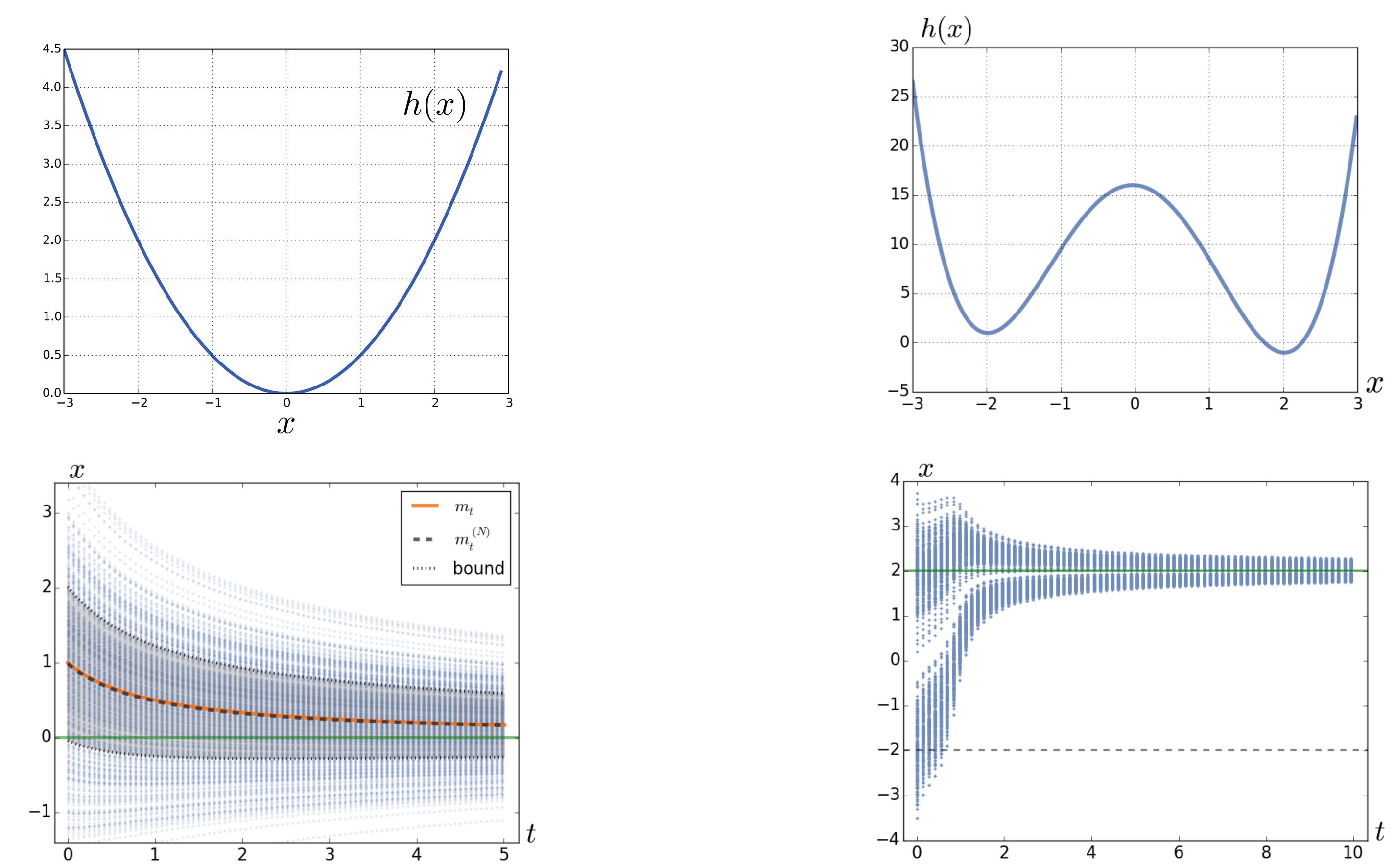
$$b_t^{(N)} = \frac{1}{N}\sum_{i=1}^N X_t^i(h(X^i) - \hat{h}_t^{(N)})$$

$$\mathrm{K}_t^{(\mathrm{N})}\Sigma_t^{(\mathrm{N})} + \Sigma_t^{(\mathrm{N})}\mathrm{K}_t^{(\mathrm{N})} = \frac{1}{N}\sum_{i=1}^N (X_t^i - m_t^{(N)})(X_t^i - m_t^{(N)})^T(h(X^i) - \hat{h}_t^{(N)})$$

Numerical simulation

Quadratic function

Double-well function



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