Data-Driven Nonlinear Filtering Algorithms with Optimal Transport Maps

Presented at the SIAM Conference on Computational Science and Engineering (CSE25)

Amirhossein Taghvaei

Joint work with Mohammad Al-Jarrah, Niyizhen Jin, and Bamdad Hosseini

Department of Aeronautics & Astronautics
University of Washington, Seattle

March 4, 2025

This talk

References:

- *Data-Driven Approximation of Stationary Nonlinear Filters with Optimal Transport Maps*
  Mohammad Al-Jarrah, Bamdad Hosseini, Amirhossein Taghvaei
  IEEE Conference on Decision and Control (CDC), Milan, 2024
- *Nonlinear Filtering with Brenier Optimal Transport Maps*
  Mohammad Al-Jarrah, Niyizhen Jin, Bamdad Hosseini, Amirhossein Taghvaei
  International Conference of Machine Learning (ICML), Vienna, 2024
- Conditional Optimal Transport on Function Spaces
  Bamdad Hosseini, Alexander W. Hsu, Amirhossein Taghvaei
  SIAM/ASA Journal on Uncertainty Quantification (Accepted)
- *Optimal Transport Particle Filters*
  Mohammad Al-Jarrah, Amirhossein Taghvaei, Bamdad Hosseini
  IEEE Conference on Decision and Control (CDC), Singapore, 2023
- *An optimal transport formulation of Bayes' law for nonlinear filtering algorithms*
  Amirhossein Taghvaei, Bamdad Hosseini
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nonlinear filtering $\xrightarrow{\text{Optimal Transport}}$ machine learning

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$$\text{nonlinear filtering} \xrightarrow{\text{Optimal Transport}} \text{machine learning}$$

Outline

- **Part I:** Nonlinear filtering problem
- **Part II:** Optimal transport filter
- **Part III:** Extension to data-driven setting

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Nonlinear filtering problem

Model:

$$X_t \sim a(\cdot \mid X_{t-1}), \quad X_0 \sim \pi_0$$
$$Y_t \sim h(\cdot \mid X_t)$$

- X_t is the state
- Y_t is the observation
- dynamic and observation models are available as simulators

Nonlinear filtering: numerical approximation of the posterior $\pi_t = P_{X_t|Y_1,\ldots,Y_t}$ for all t.

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Filtering equations

- $\pi_t := \mathsf{P}(X_t | Y_{1:t})$
- Two important operations:

$$\text{Propagation:} \quad \pi \xrightarrow{\text{dynamics}} \mathcal{A}\pi$$

$$\text{Conditioning:} \quad \pi \xrightarrow{\text{Bayes law}} \mathcal{B}_y(\pi)$$

- Recursive update law for the posterior

$$\pi_{t-1} \xrightarrow{\text{dynamics}} \pi_{t|t-1} := \mathcal{A}\pi_{t-1} \xrightarrow{\text{Bayes law}} \pi_t = \mathcal{B}_{Y_t}(\pi_{t|t-1})$$

Challenge: numerical implementation of the Bayes' law

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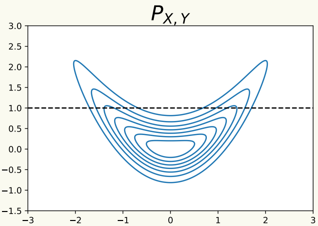
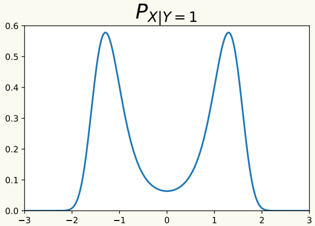
- $X \sim \mathcal{N}(0,1)$
- $Y = \frac{1}{2}X^2 + \epsilon W$
- $P_{X|Y=1} = ?$

EnKF:





G. Evensen. "Data Assimilation. The Ensemble Kalman Filter" (2006)
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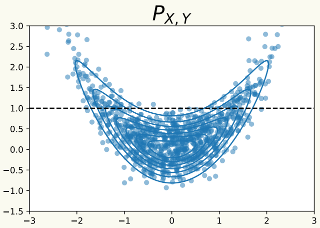

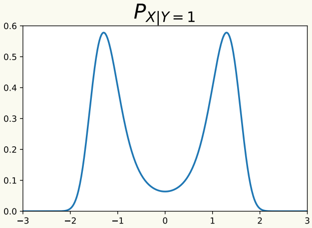

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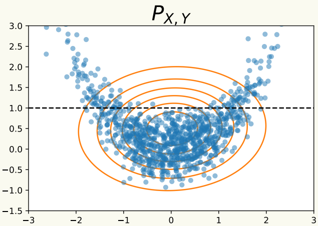

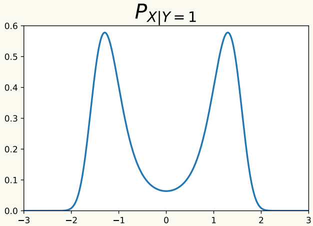

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Illustrative example
Ensemble Kalman filter (EnKF)

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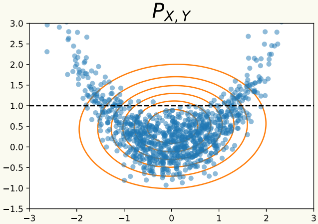

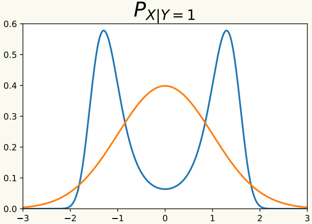

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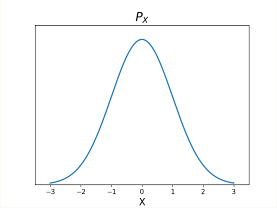
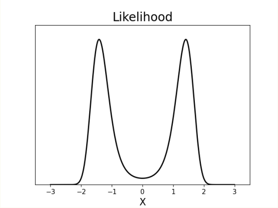
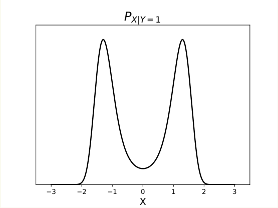
Importance sampling (IS):







small noise regime: $\epsilon \to 0$

This is the main reason for the curse of dimensionality of IS-based particle filters

P. Del Moral, A.Guionnet. On the stability of interacting processes with applications to filtering and genetic algorithms. (2001)
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Illustrative example
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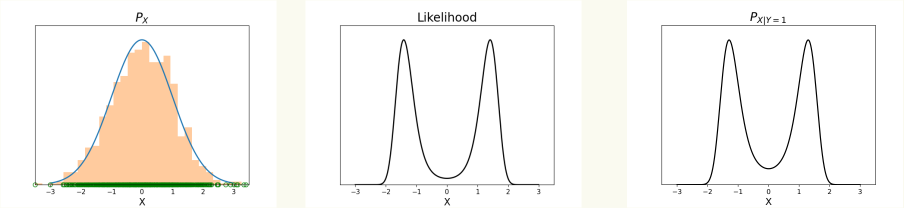

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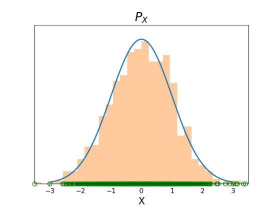

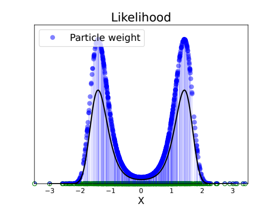

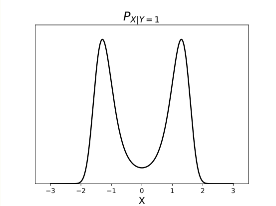

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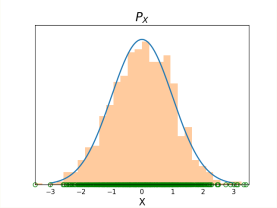

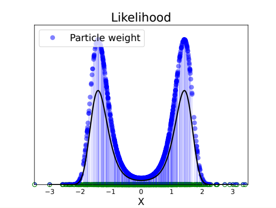

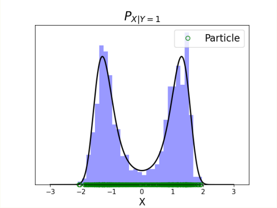

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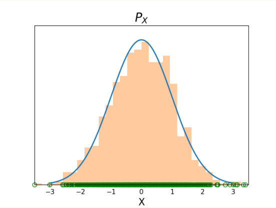

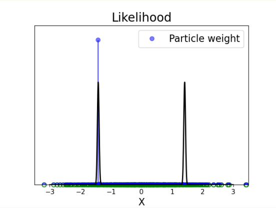

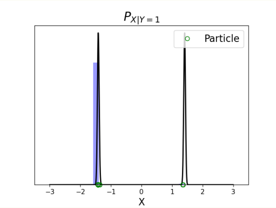

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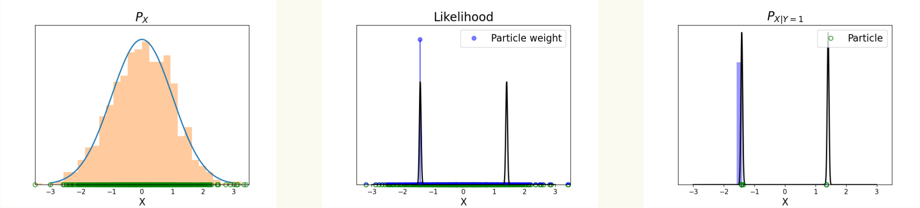

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Control and coupling techniques

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- Sampling via measure transport: An introduction [Marzouk,et. al. 2016]
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This talk: Conditioning with optimal transport map

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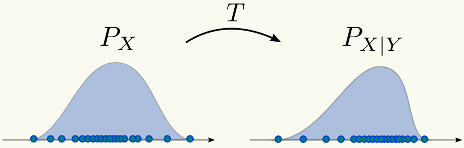
Conditioning with transport maps



$$X^i \sim P_X \longrightarrow T(X^i, y) \sim P_{X|Y=y}$$

Example:

Questions: In a general setting,

Y. Marzouk, T. Moselhy, M. Parno, A. Spantini, Sampling via measure transport: An introduction (2016)
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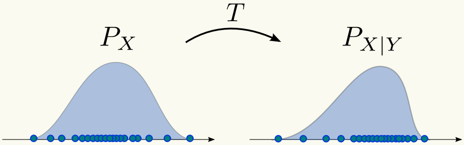

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Example:

- Consider $Y = X$. Then, $P_{X|Y=y} = \delta_y$ is represented by the map $T(x,y) = y$
- Consider jointly Gaussian (X, Y). Then $P_{X|Y=y}$ is represented by the (stochastic) map $X \mapsto X + K(y - Y)$

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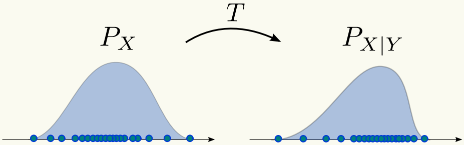

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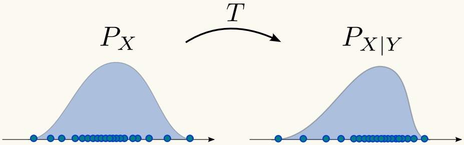

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Questions: In a general setting,

- does the map exists?
- how to numerically find it?

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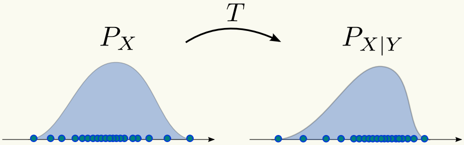
Conditioning with transport maps



$$X^i \sim P_X \longrightarrow T(X^i, y) \sim P_{X|Y=y}$$

Example:

- Consider $Y = X$. Then, $P_{X|Y=y} = \delta_y$ is represented by the map $T(x, y) = y$
- Consider jointly Gaussian (X, Y). Then $P_{X|Y=y}$ is represented by the (stochastic) map $X \mapsto X + K(y - Y)$

Questions: In a general setting,

- does the map exists?
- how to numerically find it?

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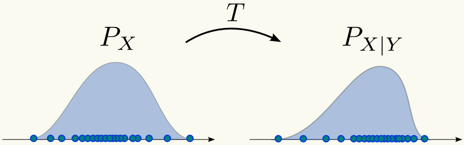

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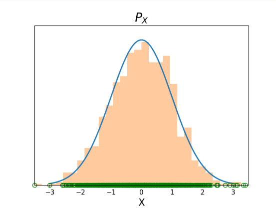
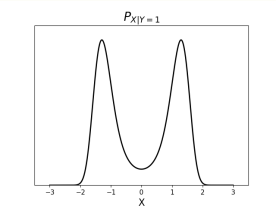
Conditioning with optimal transport map

Illustrative example

$\rightarrow$



$\xrightarrow{?}$



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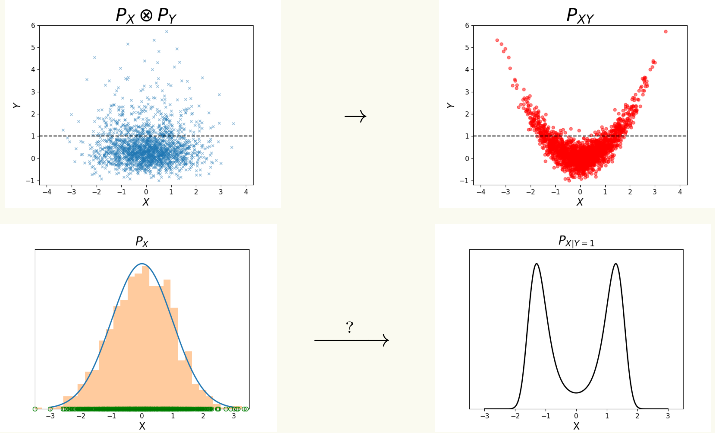

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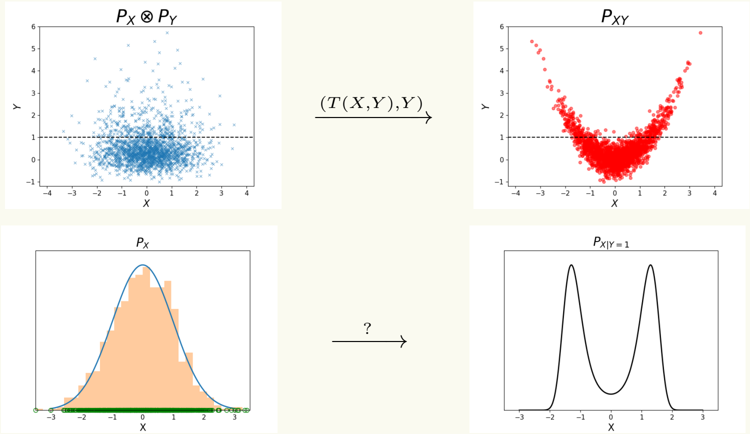

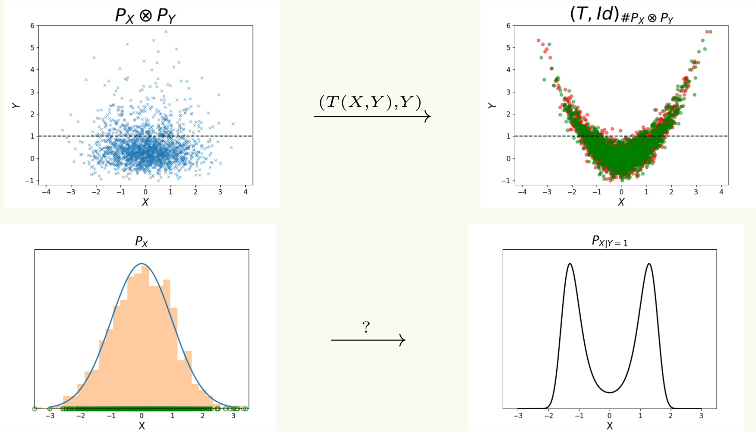

$P_X \otimes P_Y$
Y
X
$(T(X,Y),Y)$
$(T, Id)_{\#P_X \otimes P_Y}$
Y
X
P_X
X
?
$P_{X|Y=1}$
X

Conditioning with optimal transport map
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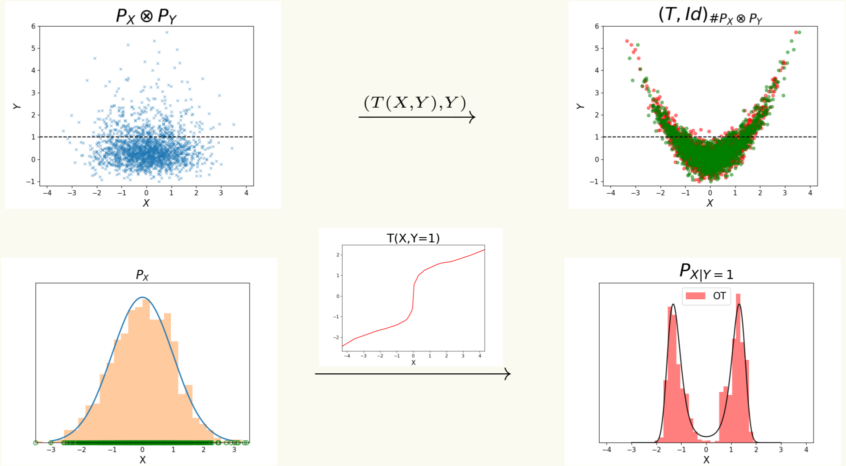

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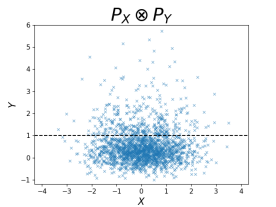
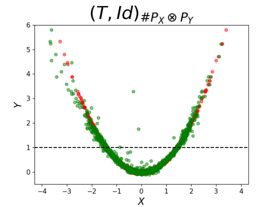
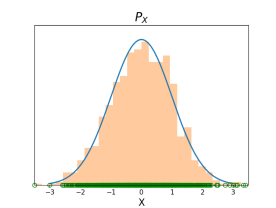
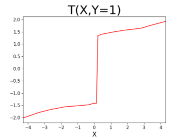
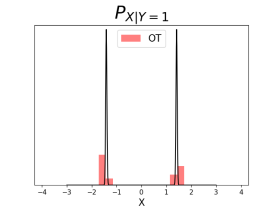


$$\xrightarrow{(T(X,Y),Y)}$$









small noise limit

Conditioning with optimal transport map

Variational formulation of the Bayes' law

Bayes law: $$P_{X|Y} = \frac{P_X P_{Y|X}}{P_Y}$$

$$= T(\cdot; Y)\#P_X$$

Conditional Kantorovich semi-dual formulation:

$$\max_{f \in c\text{-concave}_x} \min_T \mathbb{E}\left[\frac{1}{2}\|T(\bar{X}, Y) - \bar{X}\|^2 - f(T(\bar{X}, Y), Y) + f(X; Y)\right]$$

Computational properties:

- Only requires samples $(X_i, Y_i) \sim P_{XY}$ (data-driven/simulation based)
- Enables construction of "approximate" posterior distributions
- Allows application of ML tools (stochastic optimization and neural nets)

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Optimal Transport Filter

Algorithm

Initialize:

- particles $\{X_0^i\}_{i=1}^N \sim \pi_0$
- neural nets f, T

For $t = 1$ to $t = T$ do:

- propagation: $X_{t|t-1}^i \sim a(\cdot|X_{t-1}^i)$ and $Y_{t|t-1}^i \sim h(\cdot|X_{t|t-1}^i)$
- optimization: $(\hat{T}_t, \hat{f}_t) \leftarrow \max_{f \in \mathcal{F}} \min_{T \in \mathcal{T}} J(f, T; \frac{1}{N}\sum_{i=1}^N \delta_{(X_{t|t-1}^i, Y_{t|t-1}^i)})$
- conditioning: $X_t^i = \hat{T}_t(X_{t|t-1}^i, Y_t)$

Remarks:

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Outline

- **Part I:** Nonlinear filtering problem
- **Part II:** Optimal transport filter
- **Part III:** Extension to data-driven setting

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Problem setup:

$$X_t \sim a(\cdot \mid X_{t-1}), \quad X_0 \sim \pi_0$$
$$Y_t \sim h(\cdot \mid X_t)$$

- X_t is the state
- Y_t is the observation
- the dynamic and observation models are unknown

Objective:

given: $\{X_0^j, (X_1^j, Y_1^j), \ldots, (X_{t_f}^j, Y_{t_f}^j)\}_{j=1}^J$

compute: $\pi_t := P(X_t | Y_t, \ldots, Y_1), \quad \forall t \geq 0$

for a new set of observations $\{Y_t, \ldots, Y_1\}$

Nonlinear filtering problem
Data-driven setting

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Solution approach

- Exact posterior:

$$\pi_t := \mathbb{P}_{X_0 \sim \pi_0}(X_t | Y_t, \ldots, Y_1)$$

- Step 1: Truncated posterior

$$\pi^{\mu}_{t,s} := \mathbb{P}_{X_s \sim \mu}(X_t | Y_t, \ldots, Y_{s+1})$$

- Step 2: OT representation

$$\pi^{\mu}_{t,s} = T(\cdot, Y_t, \ldots, Y_s) \# \mu \quad \text{where}$$
$$T \leftarrow \max_{f \in \mathcal{F}} \min_{T \in \mathcal{T}} J(f, T; P_{X_t, Y_t, \ldots, Y_{s+1}})$$

- Step 3: Stationary assumption

$$P_{X_t, Y_t, \ldots, Y_{s+1}} = P_{X_w, Y_w, \ldots, Y_1} \quad \text{where} \quad w := t - s$$

- Step 4: Use training data to approximate $P_{X_w, Y_w, \ldots, Y_1}$

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Optimal transport data-driven filter (OT-DDF)

Offline stage:

- **Input:** Recorded data $\{X_0^j, (X_1^j, Y_1^j), \ldots, (X_w^j, Y_w^j)\}_{j=1}^J$
- Obtain map $\hat{T}_w$ by solving

$$\max_{f \in \mathcal{F}} \min_{T \in \mathcal{T}} J(f, T, \text{Data})$$

- **Output:** The map $\hat{T}_w$

Online stage:

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Online stage:

- **Input:** Map $\hat{T}_w$ and initial particles $X_0^i \sim \pi_0$
- Update particles $X_t^i = \hat{T}_w(X_0^i, Y_t, Y_{t-1}, \ldots, Y_{t-w})$
- **Output:** Particles $\{X_t^i\}_{i=1}^N, \quad \forall\, t \geq w+1$

Numerical example: Dynamical model

$$X_t = \begin{bmatrix} \alpha & \sqrt{1-\alpha^2} \\ -\sqrt{1-\alpha^2} & \alpha \end{bmatrix} X_{t-1} + \sigma V_t, \quad \alpha = 0.9, \sigma = 0.1$$

$$Y_t = h(X_t) + \sigma W_t$$

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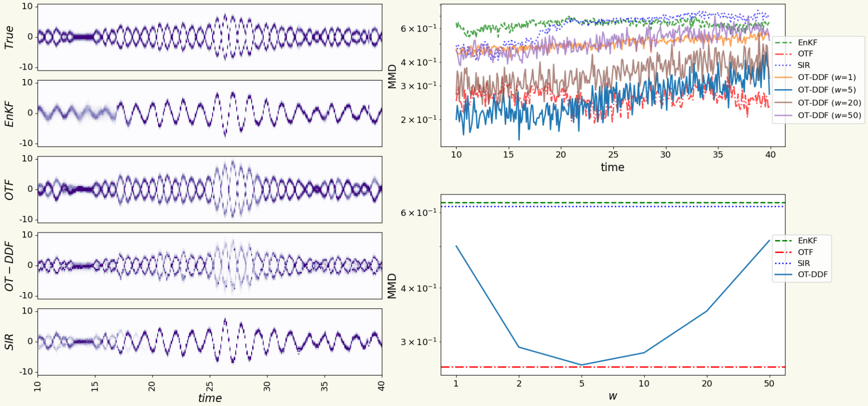
$$h(X_t) = X_t^2(1)$$



Numerical example: Lorenz 63

$$\begin{bmatrix} \dot{X}(1) \\ \dot{X}(2) \\ \dot{X}(3) \end{bmatrix} = \begin{bmatrix} \sigma(X(2) - X(1)) \\ X(1)(\rho - X(3)) - X(2) \\ X(1)X(2) - \beta X(3) \end{bmatrix}, \quad X_0 \sim \mathcal{N}(0, 10 \cdot I_3),$$

$$Y_t = h(X_t) + \sigma W_t, \quad \sigma^2 = 1$$

Numerical example: Lorenz 63

$$\begin{bmatrix} \dot{X}(1) \\ \dot{X}(2) \\ \dot{X}(3) \end{bmatrix} = \begin{bmatrix} \sigma(X(2) - X(1)) \\ X(1)(\rho - X(3)) - X(2) \\ X(1)X(2) - \beta X(3) \end{bmatrix}, \quad X_0 \sim \mathcal{N}(0, 10 \cdot I_3),$$

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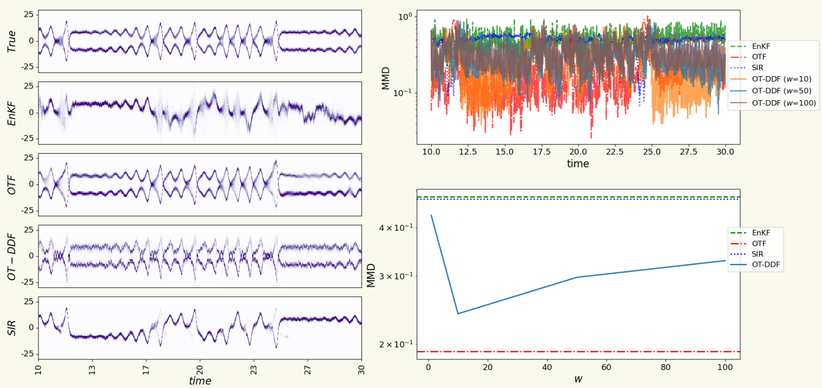
$$h(X_t) = X_t(3)$$



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Offline training time: 46.29 seconds

One-time step update:

Method	EnKF	SIR	OTF	OT-DDF
time(sec)	1.7×10^{-4}	2.0×10^{-4}	6.8×10^{-2}	1.5×10^{-4}

Error analysis

Main result

Assume

- The exact filter is exponentially stable
- The process (X_t, Y_t) is stationary
- (f, T) is a possibly non-optimal pair with max-min gap $\epsilon(f, T)$
- The function $x \mapsto \frac{1}{2}\|x\|^2 - f(x, y_w, \ldots, y_1)$ is α-strongly convex for all $(y_w, \ldots, y_1)$.

Then,

$$d(T(\cdot, Y_t, \ldots, Y_{t-w})\#\mu, \pi_t) \leq C\lambda^w M + \sqrt{\frac{4}{\alpha}\epsilon(f, T)}$$

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Acknowledgments

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References: