Numerical Methods to Solve the Weighted Poisson Equation

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Problem statement

Weighted Poisson equation

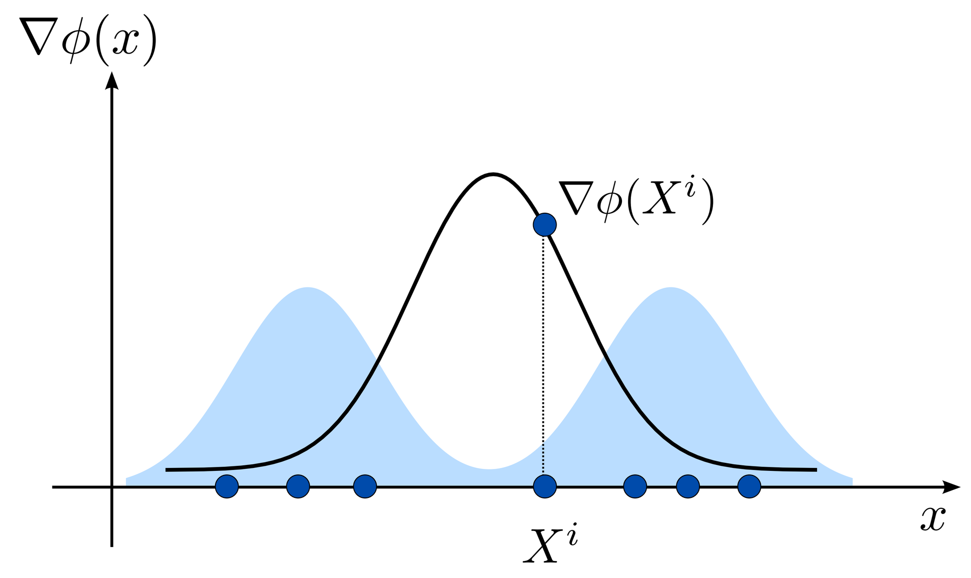
$$-\frac{1}{\rho(x)}\nabla\cdot(\rho(x)\nabla\phi(x)) = h(x) - \hat{h}$$

- $\rho : \mathbb{R}^d \to \mathbb{R}^+$ (prob. density)
- $h : \mathbb{R}^d \to \mathbb{R}$ (given function), $\hat{h} := \int h(x)\rho(x)\,\mathrm{d}x$
- $\phi : \mathbb{R}^d \to \mathbb{R}$ (solution)

Problem:

Given: $\{X^1, \ldots, X^N\} \overset{\text{i.i.d}}{\sim} \rho$

Find: $\{\nabla\phi(X^1), \ldots, \nabla\phi(X^N)\}$

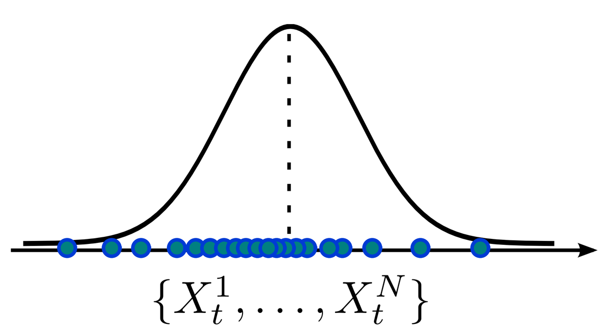


Motivation: Nonlinear filtering

Ensemble Kalman filter (EnKF)

$$\mathrm{d}X_t^i = \underbrace{AX_t^i\,\mathrm{d}t + \mathrm{d}B_t^i}_{\text{propagation}} + \underbrace{\mathsf{K}_t\,\mathrm{d}I_t^i}_{\text{update}}$$

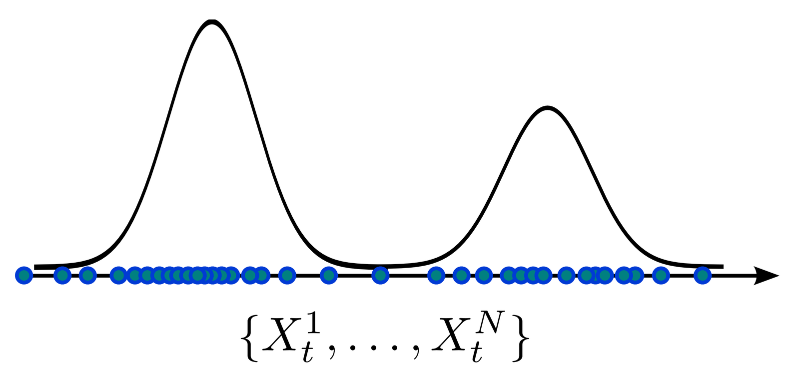
K_t is the Kalman gain



Feedback particle filter (FPF)

$$\mathrm{d}X_t^i = \underbrace{a(X_t^i)\,\mathrm{d}t + \mathrm{d}B_t^i}_{\text{propagation}} + \underbrace{\mathsf{K}_t(X_t^i)\,\mathrm{d}I_t^i}_{\text{update}}$$

$\mathsf{K}_t = \nabla\phi$ where ϕ solves the Poisson eq.



- FPF is the generalization of the EnKF for nonlinear non-Gaussian setting
- Poisson equation is the generalization of the Kalman gain

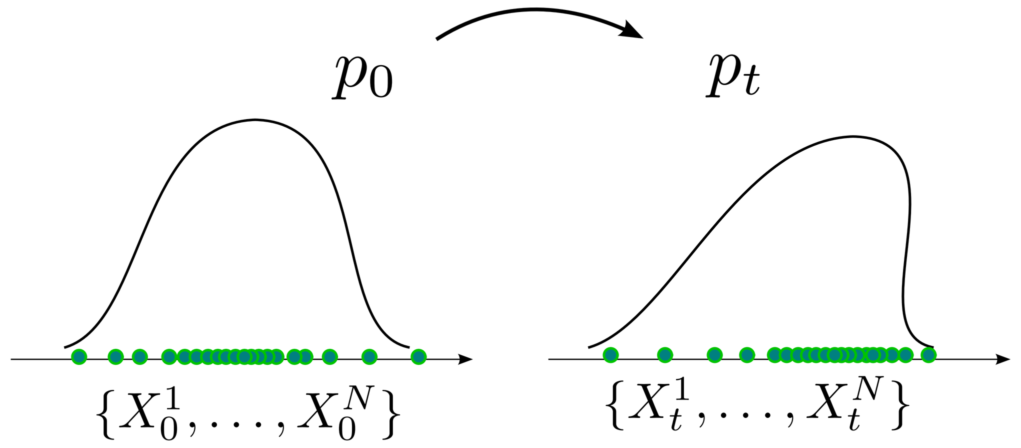
[A. Taghvaei, J. de Wiljes, P. G. Mehta, and S. Reich, Kalman filter and its modern extensions for the continuous-time nonlinear filtering problem, ASME, 2018 (To Appear)]

Motivation: Homotopy methods and optimal transportation

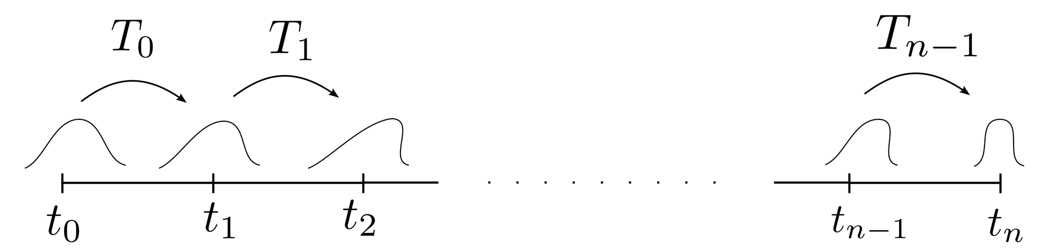
Problem

Given: $\{X_0^1, \ldots, X_0^N\} \overset{\text{i.i.d}}{\sim} p_0$ (prior)

Generate: $\{X_t^1, \ldots, X_t^N\} \overset{\text{i.i.d}}{\sim} p_t \propto p_0 e^{-h}$ (posterior)



Solution



- Construct a discrete time process:

$$X_{n+1}^i = T_n(X_n^i)$$

where T_n is the opt. transp. map between $p_n \to p_{n+1}$

- Take the continuous-time limit:

$$\frac{\mathrm{d}X_t^i}{\mathrm{d}t} = \nabla\phi(X_t^i)$$

where ϕ solves the Poisson eq.

Daum and Huang (2010-); Moselhy and Marzouk (2012); Reich (2013); Heng, Doucet and Pokern (2015)

[A. Taghvaei, P. G. Mehta, Optimal transport formulation of the feedback particle filter, ACC, 2016]

Method: Two viewpoints, Two algorithms

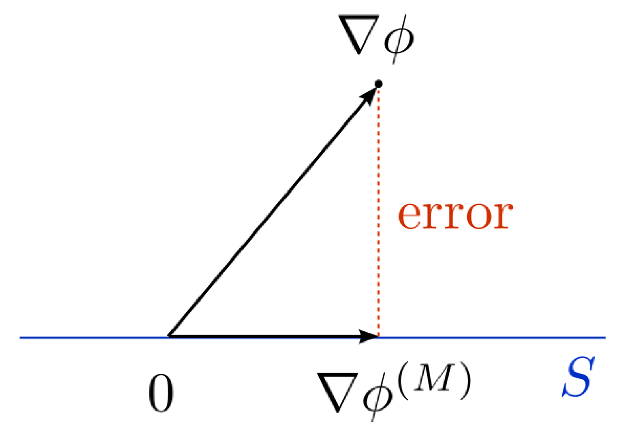
1) PDE viewpoint:

$$\langle\nabla\phi, \nabla\psi\rangle = \langle\psi, h - \hat{h}\rangle \quad \forall\psi\in H^1(\rho)$$

where $\langle f, g\rangle := \int f(x)g(x)\rho(x)\,\mathrm{d}x$

Galerkin algorithm:

- Select basis functions $\{\psi_1, \ldots, \psi_M\}$
- Express $\phi(x) = c_1\psi_1(x) + \ldots c_M\psi_M(x)$
- Solve system of M linear equations for c



2) Stochastic viewpoint:

$$\phi = P_\epsilon\phi + \int_0^\epsilon P_s(h - \hat{h})\,\mathrm{d}s$$

where $\{P_t\}$ is the semigroup for $\Delta_\rho := \frac{1}{\rho}\nabla\cdot(\rho\nabla)$

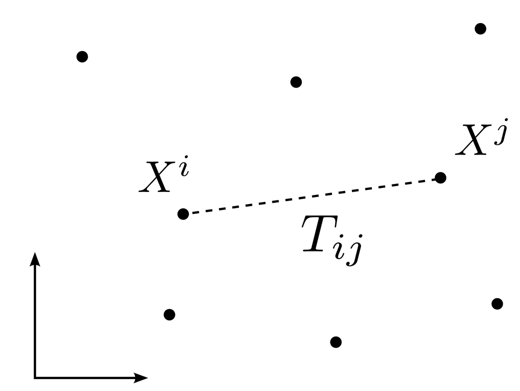
Kernel-based based algorithm:

- Approximate P with a Markov matrix

$$T_{ij} = k_\epsilon(X^i, X^j), \quad \text{for} \quad i, j = 1, \ldots, N$$

where k_ϵ is the diffusion map kernel [Coifman (2006)]

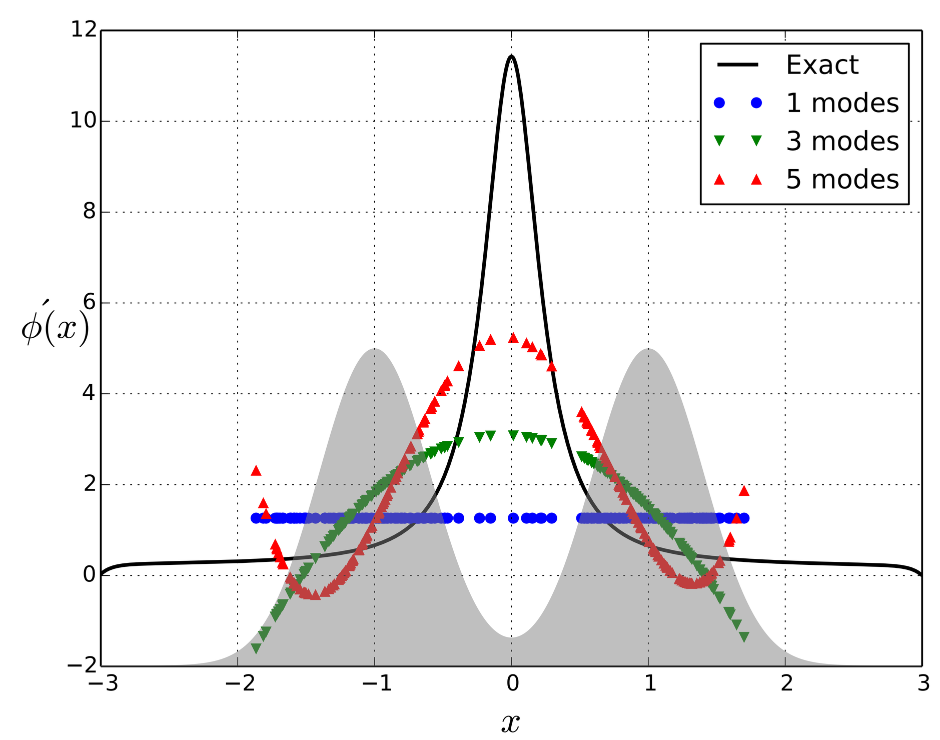
- Solve the fixed point equation $\phi = T\phi + \epsilon h$ iteratively



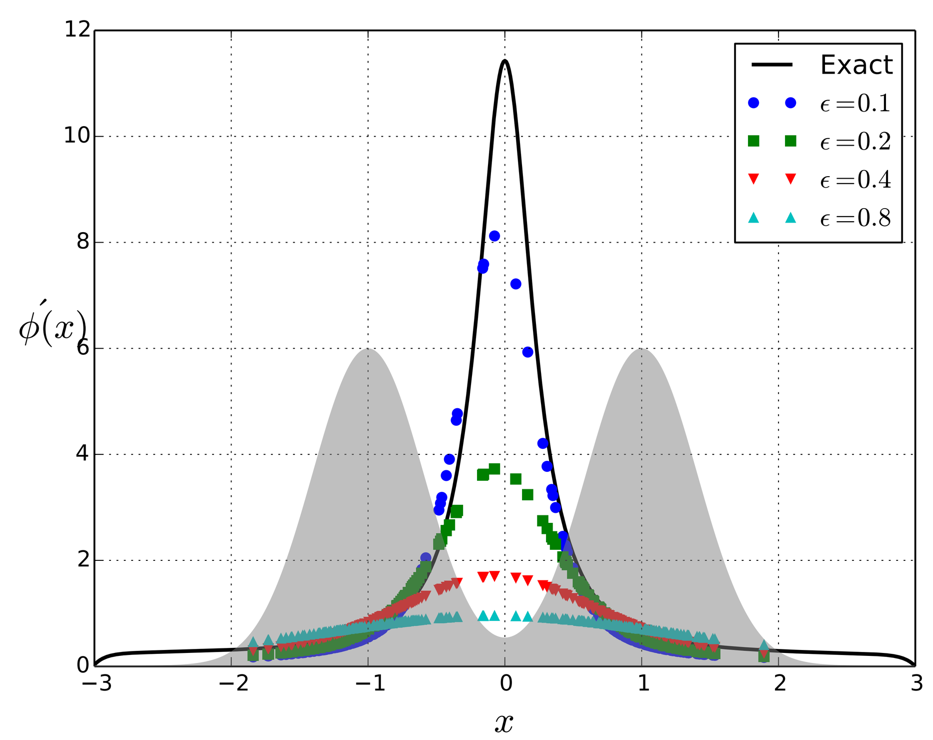
[A. Taghvaei, P. G. Mehta, Gain function approximation in the feedback particle filter, CDC, 2016]

Numerical result

Galerkin Algorithm



Kernel-based Algorithm



Error analysis

Galerkin Algorithm

$$\text{Total error} \leq \underbrace{C\|h - \Pi_S h\|_{L^2}}_{\text{Bias}} + \underbrace{\frac{1}{\sqrt{N}}\|h\|_\infty\sqrt{\sum_{m=1}^M\frac{1}{\lambda_m}}}_{\text{Variance}}$$

Kernel-based Algorithm

$$\text{Total error} \leq \underbrace{O(\epsilon)}_{\text{Bias}} + \underbrace{O\left(\frac{1}{\epsilon^{1+d/2}\sqrt{N}}\right)}_{\text{Variance}}$$

[A. Taghvaei, P. G. Mehta, S. P. meyn, Error Estimates for the Kernel Gain Function Approximation in the Feedback Particle Filter, ACC, 2017]

Acknowledgement

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