Gain Function Approximation in the Feedback Particle Filter

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Joint work with P. G. Mehta

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May 1, 2016

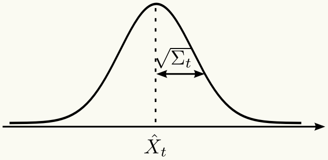
ILLINOIS

Kalman Filter:

$$\mathrm{d}X_t = AX_t\,\mathrm{d}t + \mathrm{d}B_t$$
$$\mathrm{d}Z_t = HX_t\,\mathrm{d}t + \mathrm{d}W_t$$

$\mathsf{P}(X_t|\mathcal{Z}_t) =$ Gaussian $N(\hat{X}_t, \Sigma_t)$,

$$\mathrm{d}\hat{X}_t = A\hat{X}_t\,\mathrm{d}t + \mathsf{K}_t(\mathrm{d}Z_t - H\hat{X}_t\,\mathrm{d}t)$$
$$\frac{\mathrm{d}\Sigma_t}{\mathrm{d}t} = \ldots \text{(Riccati equation)}$$



Feedback Particle Filter:

$$\mathrm{d}X_t = a(X_t)\,\mathrm{d}t + \mathrm{d}B_t$$
$$\mathrm{d}Z_t = h(X_t)\,\mathrm{d}t + \mathrm{d}W_t$$

$\mathsf{P}(X_t|\mathcal{Z}_t) \approx$ empirical dist. of $\{X^1, \ldots, X^N\}$,

$$\mathrm{d}X_t^i = a(X_t^i)\,\mathrm{d}t + \mathrm{d}B_t^i + \mathsf{K}_t(X_t^i) \circ (\mathrm{d}Z_t - \frac{h(X_t^i) + \hat{h}_t}{2}\,\mathrm{d}t)$$

Challenge: Compute the gain function $\mathsf{K}_t(.)$

T. Yang, R. S. Laugesen, P. G. Mehta, and S. P. Meyn. Multivariable feedback particle filter, *Automatica*, 2015

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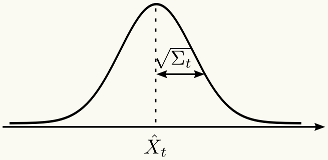

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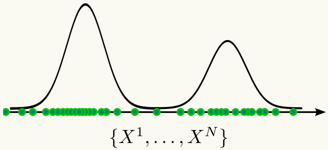

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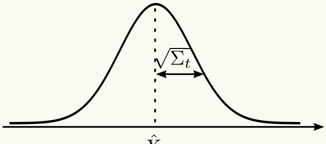

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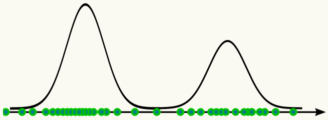

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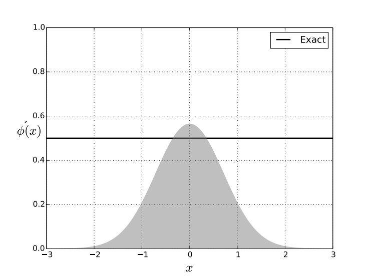
Gaussian distribution
Linear observation



$\mathsf{K}_t(x) = \text{constant}$ (Kalman gain)

Non-Gaussian distribution
Nonlinear observation

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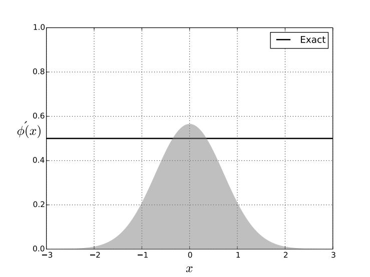

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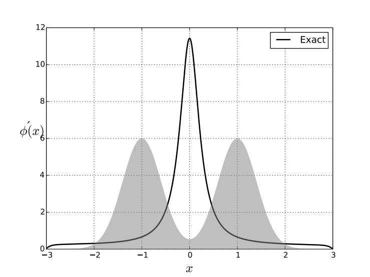

$\mathsf{K}_t(x) = \ldots$ (Nonlinear gain)

Gain function $\mathsf{K}_t(x) := \nabla\phi(x)$ where ϕ satisfies

Poisson equation: $$-\frac{1}{\rho(x)}\nabla\cdot(\rho(x)\nabla\phi(x)) = h(x) - \hat{h}$$

- ρ is a probability density function
- h is a real-valued function, $\hat{h} = \int h\rho\,\mathrm{d}x$

Poisson equation also appears in:

- Simulation and optimization theory for Markov models [Meyn, Tweedie, 2012]
- Other filtering algorithms [Daum, et. al. 2010]

Problem:

Given: $\{X^1,\ldots,X^N\} \overset{\text{i.i.d}}{\sim} \rho$

Find: $\{\nabla\phi(X^1),\ldots,\nabla\phi(X^N)\}$ (approximately)

Related work: [Berntorp, et. al. 2016],[Radhakrishnan, et. al. 2016]

R. S. Laugesen, P. G. Mehta, S. P. Meyn, and M. Raginsky. Poisson Equation in Nonlinear Filtering. SICON, 2015

Gain Function Approximation in FPF
Problem formulation

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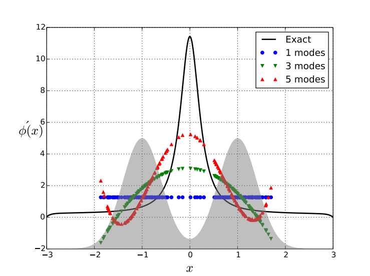
Method I: Galerkin Approximation

1. Write ϕ as linear combination of basis functions

$$\phi = c_1\psi_1 + \ldots + c_M\psi_M$$

2. Construct an M-dimensional approximation of the Poisson equation
3. Solve the system of M linear equations for $c = [c_1, \ldots, c_M]$



Issues:

- Choice of basis functions
- Gibbs phenomenon

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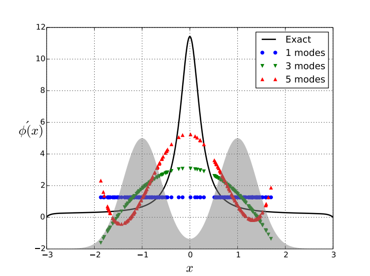

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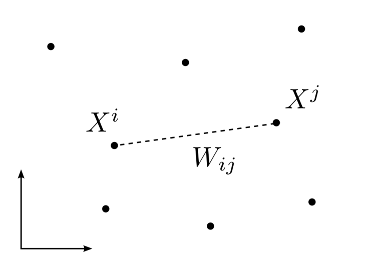
Method II: Kernel Approximation
Basic idea



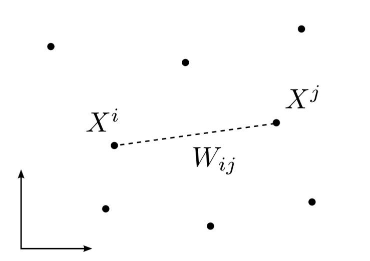
Data: $\{X^1, \ldots, X^N\} \overset{\text{i.i.d}}{\sim} \rho$

Graph Laplacian: $L := \frac{1}{\epsilon}(I - D^{-1}W), \quad W_{ij} = k_\epsilon(X^i, X^j)$

Asymptotics of the graph Laplacian: [Belkin, 2003], [Coifman, Lafon, 2006], [Hein, 2007]

$$L\phi \;\to\; -\frac{1}{\rho}\nabla\cdot(\rho\nabla\phi) \quad \text{as} \quad N\to\infty, \epsilon\to 0$$

Leads to discretization of the Poisson equation



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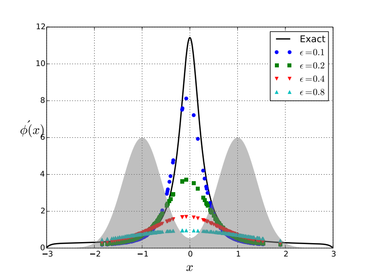
Leads to discretization of the Poisson equation

- Form an N-dimensional approximation of the Poisson equation,

$$-\frac{1}{\rho}\nabla\cdot(\rho\nabla\phi) = h \quad \approx \quad \begin{bmatrix} & & \\ & L_{(X^i,X^j)} & \\ & & \end{bmatrix}\begin{bmatrix} \phi_{(X^1)} \\ \vdots \\ \phi_{(X^N)} \end{bmatrix} = \begin{bmatrix} h_{(X^1)} \\ \vdots \\ h_{(X^N)} \end{bmatrix}$$

- Solve for $\{\phi(X^1),\ldots,\phi(X^N)\}$



Issue: Computational cost (N is large) $\to$ Use sparsity of L (future work)

A. Taghvaei, and P. G. Mehta, Gain Function Approximation in the Feedback Particle Filter, *CDC*, 2016 (submitted)

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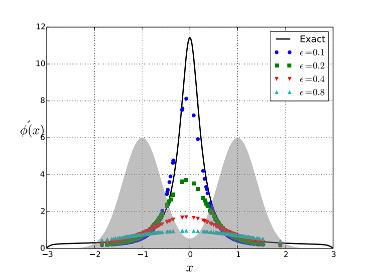

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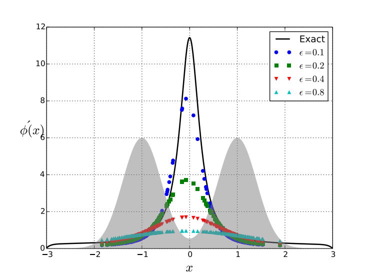

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Final slide

Thank you!

Method II: Kernel Approximation

Idea behind the proof

Weighted **Heat equation:** $\Delta f(t,x) = \frac{\partial f(t,x)}{\partial t}$ **(I)**, $f(0,x) = \phi(x)$ **(II)**

Weighted **Kernel solution:** $f(t,x) = \int g(t,x,y) f(0,y)\, \mathrm{d}y$ **(III)**

Kernel approximation of $\Delta \phi$

$$\text{(I)} \quad \Rightarrow \quad \Delta f(0,x) \approx \frac{1}{t}(f(t,x) - f(0,x))$$

$$\overset{\text{(III)}}{\Rightarrow} \quad \Delta f(0,x) \approx \frac{1}{t}\Big(\int g(t,x,y) f(0,y)\, \mathrm{d}y - f(0,x)\Big)$$

$$\overset{\text{(II)}}{\Rightarrow} \quad \Delta \phi(x) \approx \frac{1}{t}\Big(\int g(t,x,y) \phi(y)\, \mathrm{d}y - \phi(x)\Big)$$

Empirical approximation $\Delta \phi$

$$\Delta \phi(x) \approx \frac{1}{t}\Big(\frac{1}{N} \sum_{j=1}^{N} g(t,x,X^j) \phi(X^j) - \phi(x)\Big)$$

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Numerical Results