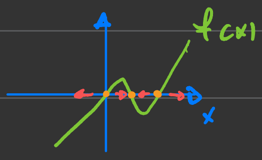
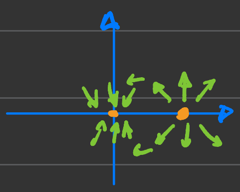
Review:

Part I: intro to nonlinear sys.

- dyn. sys. $\dot{x} = f(x)$
- eqlb. point $f(x) = 0$
- Phase portrait method → good for 1 or 2-dim systems.



1-dim. sys.



2-dim. sys.

- Linearization around eqlb. $\bar{x}$:

$$\dot{z} = Az \quad \text{where} \quad A = \frac{\partial f}{\partial x}(\bar{x})$$

explains the local behavior

$\bar{x}$ is A.S. iff A is Hurwitz

it is actually exp. stable!

- Existence and uniqueness: $\dot{x} = f(x)$, $x(0) = x_0$

• if f is Lip. on $B_r(x_0)$ $\Rightarrow$ $\exists!$ solution for $t \in [0, \delta]$

• if f is globally Lip. $\Rightarrow$ $\exists!$ solution for all t.

• if f is Lip. on D and solution never exits D $\Rightarrow$ $\exists!$ solution for all t.

- Lipschitz functions: f is Lip. on D if $\exists L > 0$ s.t.

$$\|f(x) - f(y)\| \leq L \|x - y\|$$

check Lip. by showing $\|\frac{\partial f}{\partial x}(x)\| \leq L$ $\forall x \in W$ convex set.

- perturbation analysis:

$$\dot{x} = f(t, x) \qquad x(0) = x_0$$
$$\dot{y} = f(t, y) + g(t, y) \qquad y(0) = y_0$$

① f is globally Lip. (with const. L)

② g is uniformly bounded (with cons. M)

$$\Rightarrow \|x(t) - y(t)\| \leq e^{tL} \|x_0 - y_0\| + \frac{M}{L}(e^{tL} - 1)$$

- Comparison lemma: $\alpha(t) \in \mathbb{R}$

$$\begin{aligned} &\dot{\alpha}(t) \le f(t, \alpha(t)) \\ &\alpha(0) \le \beta_0 \end{aligned} \Rightarrow \alpha(t) \le \beta(t) \quad \text{where} \quad \begin{aligned} &\dot{\beta}(t) = f(t, \beta(t)) \\ &\beta(0) = \beta_0 \end{aligned}$$

- BG lemma:

$$\alpha(t) \le \int_0^t \underbrace{a(s)}_{\ge 0} \alpha(s)\, ds + \alpha_0 + \int_0^t u(s)\, ds$$

$$\Rightarrow \alpha(t) \le \phi(t,0)\, \alpha_0 + \int_0^t \underbrace{\phi(t,s)}_{e^{\int_s^t a(\tau) d\tau}} u(s)\, ds$$

- Integral representation of $\dot{x} = f(x)$

$$x(t) = x(0) + \int_0^t f(x(s))\, ds \quad \Rightarrow \text{useful for analysis}$$

Example:

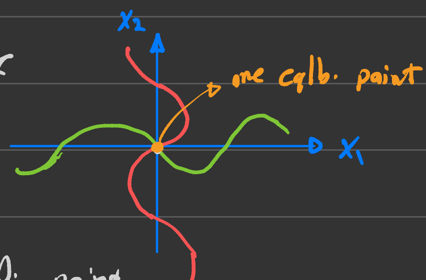
$$\dot{x} = \begin{bmatrix} -\frac{1}{2}x_1 + \frac{1}{3}\sin(x_2) \\ -\frac{1}{2}x_2 - \frac{1}{3}\sin(x_1) \end{bmatrix}$$

① find eqlb. points

$$-\frac{1}{2}x_1 + \frac{1}{3}\sin(x_2) = 0 \Rightarrow x_1 = \frac{2}{3}\sin(x_2)$$

$$-\frac{1}{2}x_2 - \frac{1}{3}\sin(x_1) = 0 \Rightarrow x_2 = -\frac{2}{3}\sin(x_1)$$

eqlb. point are the intersection of these two graphs.



$\Leftrightarrow$ $\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is the only eqlb. point

② linearization:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -\frac{1}{2} & +\frac{1}{3}\cos(x_2) \\ -\frac{1}{3}\cos(x_1) & -\frac{1}{2} \end{bmatrix} \Rightarrow \frac{\partial f}{\partial x}(\bar{x}) = \underbrace{\begin{bmatrix} -\frac{1}{2} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{2} \end{bmatrix}}_{\text{Hurwitz}}$$

$\Rightarrow$ $\bar{x} = 0$ is exp. stable.

③ is f Lip.?

$$\left\|\frac{\partial f}{\partial x}(x)\right\|_\infty \leqslant \max\left(\frac{1}{2}+|\cos(x_1)|, \frac{1}{2}+|\cos(x_2)|\right) \leqslant \frac{3}{2}$$

$\Rightarrow$ f is globally Lip.

④ does a unique sol. exist? yes, and for all $t \geq 0$.

⑤ Can we obtain a bound on $\|X(t)\|_\infty$ assuming $\|X(0)\|_\infty \leq 1$. ?

$$X(t) = X(0) + \int_0^t f(X(s)) - f(0) \, ds$$

$$\Rightarrow \underbrace{\|X(t)\|}_{\alpha(t)} \leqslant \|X(0)\| + \int_0^t \|f(X(s)) - f(0)\| \, ds$$

$$\leqslant \underbrace{1}_{\alpha_0} + \int_0^t \underbrace{L}_{a(s)} \underbrace{\|X(s) - 0\|}_{\alpha(s)} \, ds \quad , \quad u(s) = 0$$

BG
$\Rightarrow \|X(t)\| \leqslant e^{tL} = e^{\frac{3}{2}t}$

Part 2: $\dot{x} = f(x)$

- Stability analysis for an eqlb. point $\bar{x}$.

- WLOG, assume $\bar{x} = 0$ (by changing the frame)

- If you find a func. $V : \mathbb{R}^n \to \mathbb{R}$ s.t. $V(x) > 0 \ \forall x \neq 0$

① $\underbrace{\dot{V}(x)}_{\nabla V(x)^T f(x)} \leq 0 \quad \forall x \in \underbrace{D}_{\text{containing } 0} \quad \Rightarrow \quad$ stable

② $\dot{V}(x) < 0 \quad \forall x \in D, \ x \neq 0 \quad \Rightarrow \quad$ AS

③ $\dot{V}(x) < 0 \quad \forall x \in \mathbb{R}^n, \ x \neq 0 \quad \Rightarrow \quad$ GAS

and V is radially unbdd

radius of a ball inside D

$\delta = \sqrt{\frac{a_1}{a_2}}\, r$

②' $a_1 \|x\|^2 \leq V(x) \leq a_2 \|x\|^2$

$\dot{V}(x) \leq -a_3 \|x\|^2 \quad \forall x \in D \quad \Rightarrow \quad$ exp. stable

$\|x(t)\| \leq C e^{-\lambda t} \|x(0)\|$ if $\|x(0)\| < \delta$

③' $D = \mathbb{R}^n \Rightarrow$ globally exp. stable

$\|x(t)\| \leq C e^{-\lambda t} \|x(0)\| \quad \forall x(0) \in \mathbb{R}^n$

C: $\sqrt{\frac{a_2}{a_1}}$ $\qquad \lambda$: $\frac{a_3}{2a_2}$

- LaSalle's invariance principle:

  - assume $x(t) \in \Omega$ (bounded) $\forall t$ and $\dot{V}(x) \leq 0 \ \forall x \in \Omega$.

    $\Rightarrow x(t) \to M \subseteq E = \{ x \in \Omega ; \dot{V}(x) = 0 \}$

    largest invariant subset.

(1') Consider the same setting as in (1)

if the only solution that remains in E is $x(t) = 0 \ \forall t$.

then $\bar{x} = 0$ is A.S.

- Other conclusions from Lyapunov func if $\dot{V}(x) \leq 0 \ \forall x$

  $\Rightarrow$ all sub-level sets of V are invariant sets

  $\Rightarrow$ $x(t)$ is bdd if V is radially unbdd.

- Finding Lyapunov func. for lin. sys. $\dot{x} = Ax$.

$$V(x) = x^T P x \quad \text{where } P \succ 0 \text{ and solves}$$

$$A^T P + PA + \underline{Q} = 0 \quad \rightarrow \text{It has a unique p.d. solution iff } A \text{ is Hurwitz}$$

$$Q \succ 0$$

$$\Rightarrow \dot{V}(x) = -x^T Q x < 0 \quad \forall x \neq 0$$

- Application to nearly lin. sys.:

$$\dot{x} = \underbrace{Ax}_{\text{Stable}} + g(x) \longrightarrow \lim_{\|x\| \to 0} \frac{\|g(x)\|}{\|x\|} = 0$$

$\Rightarrow$ $x = 0$ is exp. stable

use $V(x) = x^T P x$ where $A^T P + PA + Q = 0$

and show $\dot{V}(x) < 0$ if $\|x\|$ is small.

- Application to gradient flow: $\dot{x} = -\nabla J(x)$

- $\min_{s \in [0,t]} \|\nabla J(x(s))\|^2 \leq \frac{c}{t}$
- $J(x(t)) - \min J \leq \frac{c}{t}$ if J is convex.
- $J(x(t)) - \min J \leq c e^{-\lambda t}$ if J is gradient dominant.
- $\|x(t) - x^*\|^2 \leq c e^{-\lambda t}$ if J is strongly convex.

<u>Example:</u>

- Consider gradient flow $\dot{x} = -\nabla J(x)$.

① assume $J(x) > c$ if $\|x\| > r_2$
and $J(x) \leq c$ if $\|x\| \leq r_1$ $\Big\}$ *

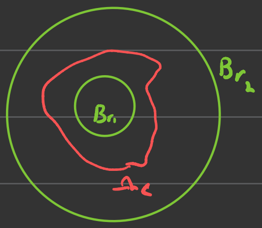
show that $\|x(t)\| \leq r_2 \ \forall t$ if $\|x(0)\| \leq r$

let $J(x)$ be candidate Lyapunov func. Then

$$\dot{J}(x) = -\|\nabla J(x)\|_2^2 \leq 0 \qquad \forall x$$

$\Rightarrow$ all sublevel sets are invariant

take $\Omega_c = \{ x \in \mathbb{R}^n ;\ J(x) \leq c \}$



then $(*) \Rightarrow B_{r_1}(0) \subseteq \Omega_c \subseteq B_{r_2}(0)$

therefore if $\|x(0)\| \leq r_1 \Rightarrow x(0) \in B_{r_1}(0) \Rightarrow x(0) \in \Omega_c \Rightarrow$

$x(t) \in \Omega_c \ \forall t \Rightarrow x(t) \in B_{r_2}(0) \Rightarrow \|x(t)\| \leq r_2$

② assume $\bar{x}$ is a local minimizer, that is to say

$$\nabla J(\bar{x}) = 0 \quad \text{and} \quad \lambda_{min}(\nabla^2 J(\bar{x})) > 0$$

show $\bar{x}$ is exp. stable.

linearization around $\bar{x}$: $\dot{z} = Az$

where $A = -\frac{\partial}{\partial x}(\nabla J)$

$= -\nabla^2 J(\bar{x})$

A is Hurwitz because

$$\lambda_{max}(A) = -\lambda_{min}(\nabla^2 J(\bar{x})) < 0$$

$\Leftrightarrow$ exp. stable.

③ how to calculate convergence region if $\nabla^2 f$ is L-Lipschitz?

Part III:

- input-output sys.

$$\dot{x} = f(x,u) \qquad u \to \boxed{H} \to y$$

$$y = h(x,u)$$

$f(0,0) = 0$, $h(0,0) = 0$

- input-output stability (with finite gain)

$$\|y\|_L \leq \gamma \|u\|_L + c$$

signal norm

smallest possible const. is gain.

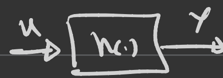
- L_2-gain of a lin. sys. $\quad \max\limits_{w} \|\underbrace{G(jw)}_{\text{transfer func.}}\|_2$

- Lyapunov method:

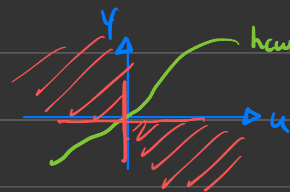
$$\dot{V} \leq \alpha^2 \|u\|^2 - \beta^2 \|y\|^2$$

$\Rightarrow$ L_2-stable with gain $\gamma \leq \frac{\alpha}{\beta}$

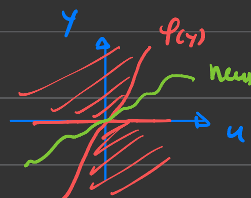
- passive memoryless systems



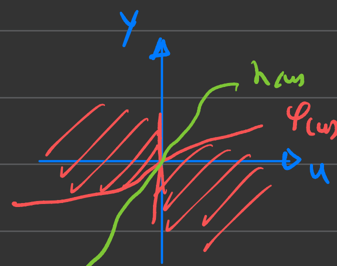
passive $\quad 0 \leq u^T y$



output s.p. $\quad 0 \leq u^T y - y^T \varphi(y)$



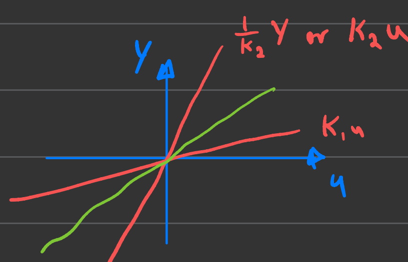
input s.p. $\quad 0 \leq u^T y - u^T \varphi(u)$



- special case : output s.p. with $\varphi(y) = \frac{1}{K_2} y \qquad K_2 > K_1$

and input s.p. with $\varphi(u) = K_1 u$

then we say h belongs to the sector $[K_1, K_2]$



- dynamic passive sys: find $V(x)$ s.t. $V(x) \geq 0 \ \forall x$
$V(0) = 0$

passive $\dot{V} \leq u^T y$

output s.p. $\dot{V} \leq u^T y - y^T \varphi(y)$

s.p. $\dot{V} \leq u^T y - W(x)$

- passivity and stability: when $u = 0$

passive $\Rightarrow$ stable

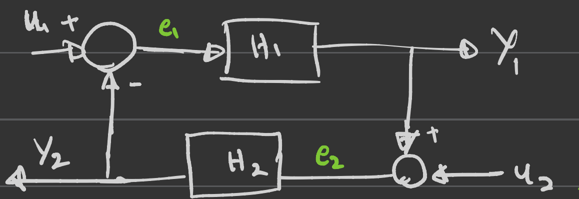
s.p. $\Rightarrow$ A.S.

output s.p. + zero-state obs $\Rightarrow$ A.S.

- passivity and input-output stability:

$$\dot{V} \leq u^T y - \delta \|y\|^2 \Longrightarrow L_2\text{-stable with } \gamma \leq \frac{1}{\delta}$$

Feedback systems:



- Small gain thm:

H_1 is L-stable with gain γ_1

H_2 " " γ_2 $\Rightarrow$ feedback sys is L-stable.

$\gamma_1 \gamma_2 < 1$

- Passivity:

- H_1 and H_2 are passive $\Rightarrow$ stable feedback sys. when $u=0$

- H_1 and H_2 are either s.p. or output s.p. + zero-state obs

$\Rightarrow$ A.S. feedback sys.

- $\dot{V}_1 \leq u_1^T y_1 - \varepsilon_1 \|u_1\|^2 - \delta_1 \|y_1\|^2$

$\dot{V}_2 \leq u_2^T y - \varepsilon_2 \|u_2\|^2 - \delta_2 \|y_2\|^2$ $\Rightarrow$ L_2-stable feedback system.

$\varepsilon_1 + \delta_2 > 0$ and $\varepsilon_2 + \delta_1 > 0$

Example:

- Consider the sys $\left.\begin{aligned}\dot{x} &= f(x,u) \\ y &= h(x)\end{aligned}\right\} H$

- Assume

① passive with radially unbdd storage func.

② Zero-state obs.

- Then $x=0$ is GAS with output feedback control

$u=-K(y)$ if $y^T K(y) > 0 \quad \forall\, y \neq 0$

- To see this, use the storage func. of H as a candidate Lyapunov func.

$$\dot{V} \overset{\text{Passivity}}{\leq} u^T y = -y^T K(y) < 0 \quad \forall\, y \neq 0$$

$\Rightarrow$ LaSalle + Zero-state obs $\Rightarrow$ A.S. $\xrightarrow{\text{radially unbdd}}$ GAS.