Part III:

- input-output sys.

$$\dot{x} = f(x,u) \qquad u \longrightarrow \boxed{H} \longrightarrow y$$
$$y = h(x,u)$$

$f(0,0)=0$, $h(0,0)=0$

- input-output stability (with finite gain)

$$\underbrace{\|y\|_L}_{\text{signal}} \leq \gamma \underbrace{\|u\|_L}_{\text{norm}} + c$$

smallest possible const. is gain.

- L_2-gain of a lin. sys. $\max_{\omega} \|\underbrace{G(j\omega)}_{\text{transfer func.}}\|_2$

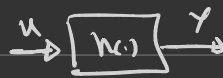
(assumes A is Hurwitz)

- Lyapunov method:

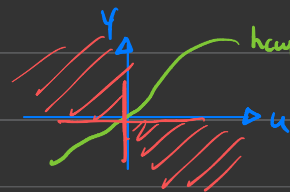
$$\dot{V} \leq \alpha^2 \|u\|^2 - \beta^2 \|y\|^2$$

$\Rightarrow$ L_2-stable with gain $\gamma \leq \frac{\alpha}{\beta}$

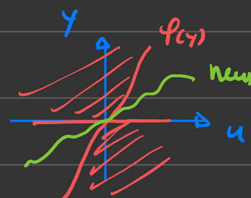
- passive memoryless systems



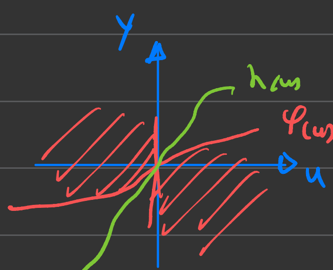
passive $\quad 0 \leq u^T y$



output s.p. $\quad 0 \leq u^T y - y^T \varphi(y)$



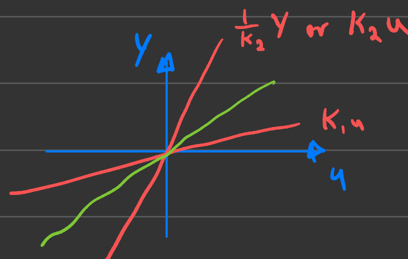
input s.p. $\quad 0 \leq u^T y - u^T \varphi(u)$



- special case : output s.p. with $\varphi(y) = \frac{1}{K_2} y \qquad K_2 > K_1$

and input s.p. with $\varphi(u) = K_1 u$

then we say h belongs to the sector $[K_1, K_2]$



- dynamic passive sys: find $V(x)$ s.t. $V(x) \geq 0 \ \forall x$
$V(0) = 0$

passive $\quad \dot{V} \leq u^T y$

output s.p. $\quad \dot{V} \leq u^T y - y^T \varphi(y)$

s.p. $\quad \dot{V} \leq u^T y - W(x)$

- passivity and stability: when $u = 0$

passive $\Rightarrow$ stable

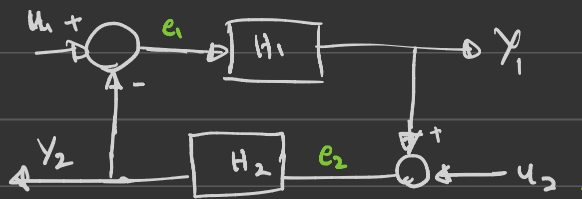
s.p. $\Rightarrow$ A.S.

output s.p. + zero-state obs $\Rightarrow$ A.S.

- passivity and input-output stability:

$$\dot{V} \leq u^T y - \delta \|y\|^2 \Longrightarrow L_2\text{-stable with } \gamma \leq \frac{1}{\delta}$$

Feedback systems:



- Small gain thm:

H_1 is L-stable with gain γ_1

and H_2 " " γ_2 $\Rightarrow$ feedback sys

and $\gamma_1 \gamma_2 < 1$ is L-stable.

- Passivity:

- H_1 and H_2 are passive $\Rightarrow$ stable feedback sys. when $u = 0$

- H_1 and H_2 are either s.p. or output s.p. + zero-state obs

$\Rightarrow$ A.S. feedback sys.

- $\dot{V}_1 \leq u_1^T y_1 - \varepsilon_1 \|u_1\|^2 - \delta_1 \|y_1\|^2$

$\dot{V}_2 \leq u_2^T y - \varepsilon_2 \|u_2\|^2 - \delta_2 \|y_2\|^2$ $\Rightarrow$ L_2-stable feedback system.

$\varepsilon_1 + \delta_2 > 0$ and $\varepsilon_2 + \delta_1 > 0$

Example:

- Consider the sys $\left.\begin{aligned}\dot{x} &= f(x,u)\\ y &= h(x)\end{aligned}\right\}H$

- Assume

① passive with radially unbdd storage func.

② Zero-state obs.

- Then $x=0$ is GAS with output feedback control

$u = -K(y)$ if $y^T K(y) > 0 \quad \forall\, y \neq 0$

- To see this, use the storage func. of H as a candidate Lyapunov func.

$$\dot{V} \overset{\text{Passivity}}{\leq} u^T y = -y^T K(y) < 0 \quad \forall\, y \neq 0$$

$\Rightarrow$ LaSalle + Zero-State obs $\Rightarrow$ A.S. $\underset{\text{radially unbdd}}{\Longrightarrow}$ GAS.

Control Lyapunov functions:

- Consider the sys.

$$\dot{x} = f(x,u)$$

assume $f(0,0)=0$

- Objective: design a control law $u = K(x)$ to make the sys. A.S.

Example:

(1) $\dot{x} = x^2 + xu, \qquad x,u \in \mathbb{R}$

let $u = K(x) = -x-1$

$\Rightarrow \dot{x} = x^2 + x(-x-1) = -x \rightarrow$ GAS.

what if we want $K(0)=0$ so that we do not input is zero when $x=0$

let $u = k(x) = -x - x^2$

$$\Rightarrow \dot{x} = x^2 - x(-x - x^2) = -x^3 \rightarrow \text{GAS.}$$

② $\dot{x} = x + x^2 u$

let $u = k(x) = \begin{cases} -\frac{2}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ → Not Continuous Control law

↓

In fact it can be shown that it is impossible to stabilize this with a Cont. Control law

$$\Rightarrow \dot{x} = -x \rightarrow \text{GAS.}$$

③ $\dot{x} = x + x^2(x-1)u$

→ impossible to stabilize because there is no control authority when $x = 1$.

→ How to design stabilizing control laws? → Lyapunov functions.

<u>Def</u>: $V: \mathbb{R}^n \to \mathbb{R}$ is a Control Lyapunov func (CLF)

for $\dot{x} = f(x,u)$ if V is p.d. and

$$\min_u \underbrace{\dot{V}(x,u)}_{\nabla V(x)^T f(x,u)} < 0 \qquad \forall x \neq 0$$

or $\forall x \neq 0, \exists u$ s.t. $\dot{V}(x,u) < 0$

- let $\Omega(x) = \{u ; \dot{V}(x,u) < 0\}$ → set of all control inputs that decrease V

- Then for any control law

s.t. $K(x) \in \Omega(x) \quad \forall x$, the sys. becomes A.S.

because $\dot{V}(x, K(x)) < 0$

- We will focus on the special class of control affine systems → we will give simple condition to check CLF conditions

→ we will give an analytical formula for $K(x)$

Control affine systems:

$$(**) \qquad \dot{x} = f(x) + g(x)\,u$$

where $x \in \mathbb{R}^n$, $f(x) \in \mathbb{R}^n$, $g(x) \in \mathbb{R}^{n\times m}$, $u \in \mathbb{R}^m$

Example: $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2^3 u_1 \\ x_2^2 + \cos(x_1) u_2 \end{bmatrix} = \underbrace{\begin{bmatrix} x_1 \\ x_2^2 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} x_2^3 & 0 \\ 0 & \cos(x_1) \end{bmatrix}}_{g(x)} \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{u}$$

Lemma:

V is a CLF for $(**)$ iff

$\forall x \neq 0$ s.t. $\nabla V(x)^T g(x) = 0$, we have $\nabla V(x)^T f(x) < 0$

Thm:

Suppose $(**)$ has a CLF. Then, $\exists$ feedback control law $K(x)$ that makes $(**)$ A.S.. And K is C^1 away from 0.

Sontag's formula for $K(x)$:

define $b(x) = \nabla V(x)^T g(x) \in \mathbb{R}^m$

$a(x) = \nabla V(x)^T f(x) \in \mathbb{R}$

$$\alpha(x) = \frac{a(x) + \sqrt{a(x)^2 + \|b(x)\|_2^4}}{\|b(x)\|_2^2}$$

then

$$K(x) = \begin{cases} -\alpha(x)\, b(x) & \text{if } b(x) \neq 0 \\ 0 & \text{else} \end{cases}$$