

2.1 A Geometric Way of Thinking

Pictures are often more helpful than formulas for analyzing nonlinear systems. Here we illustrate this point by a simple example. Along the way we will introduce one of the most basic techniques of dynamics: *interpreting a differential equation as a vector field*.

Consider the following nonlinear differential equation:

$$\dot{x} = \sin x. \quad (1)$$

To emphasize our point about formulas versus pictures, we have chosen one of the few nonlinear equations that can be solved in closed form. We separate the variables and then integrate:

$$dt = \frac{dx}{\sin x},$$

which implies

$$\begin{aligned} t &= \int \csc x \, dx \\ &= -\ln|\csc x + \cot x| + C. \end{aligned}$$

To evaluate the constant C , suppose that $x = x_0$ at $t = 0$. Then $C = \ln|\csc x_0 + \cot x_0|$. Hence the solution is

$$t = \ln \left| \frac{\csc x_0 + \cot x_0}{\csc x + \cot x} \right|. \quad (2)$$

This result is exact, but a headache to interpret. For example, can you answer the following questions?

1. Suppose $x_0 = \pi/4$; describe the qualitative features of the solution $x(t)$ for all $t > 0$. In particular, what happens as $t \rightarrow \infty$?
2. For an *arbitrary* initial condition x_0 , what is the behavior of $x(t)$ as $t \rightarrow \infty$?

Think about these questions for a while, to see that formula (2) is not transparent.

In contrast, a graphical analysis of (1) is clear and simple, as shown in Figure 2.1.1. We think of t as time, x as the position of an imaginary particle moving along the real line, and \dot{x} as the velocity of that particle. Then the differential equation $\dot{x} = \sin x$ represents a **vector field** on the line: it dictates the velocity vector \dot{x} at each x . To sketch the vector field, it is convenient to plot \dot{x} versus x , and then draw arrows on the x -axis to indicate the corresponding velocity vector at each x . The arrows point to the right when $\dot{x} > 0$ and to the left when $\dot{x} < 0$.

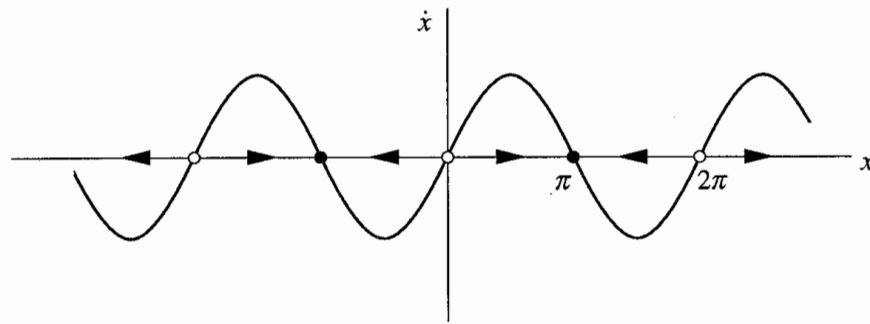


Figure 2.1.1

Here's a more physical way to think about the vector field: imagine that fluid is flowing steadily along the x -axis with a velocity that varies from place to place, according to the rule $\dot{x} = \sin x$. As shown in Figure 2.1.1, the **flow** is to the right when $\dot{x} > 0$ and to the left when $\dot{x} < 0$. At points where $\dot{x} = 0$, there is no flow; such points are therefore called **fixed points**. You can see that there are two kinds of fixed points in Figure 2.1.1: solid black dots represent **stable** fixed points (often called *attractors* or *sinks*, because the flow is toward them) and open circles represent **unstable** fixed points (also known as *repellers* or *sources*).

Armed with this picture, we can now easily understand the solutions to the differential equation $\dot{x} = \sin x$. We just start our imaginary particle at x_0 and watch how it is carried along by the flow.

This approach allows us to answer the questions above as follows:

1. Figure 2.1.1 shows that a particle starting at $x_0 = \pi/4$ moves to the right faster and faster until it crosses $x = \pi/2$ (where $\sin x$ reaches its maximum). Then the particle starts slowing down and eventually approaches the stable fixed point $x = \pi$ from the left. Thus, the qualitative form of the solution is as shown in Figure 2.1.2.

Note that the curve is concave up at first, and then concave down; this corresponds to the initial acceleration for $x < \pi/2$, followed by the deceleration toward $x = \pi$.

2. The same reasoning applies to any initial condition x_0 . Figure 2.1.1 shows that if $\dot{x} > 0$ initially, the particle heads to the right and asymptotically approaches the nearest stable fixed point. Similarly, if $\dot{x} < 0$ initially, the particle approaches the nearest stable fixed point to its left. If $\dot{x} = 0$, then x remains constant. The qualitative form of the solution for any initial condition is sketched in Figure 2.1.3.

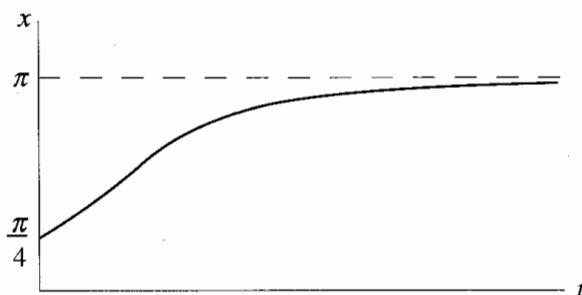


Figure 2.1.2

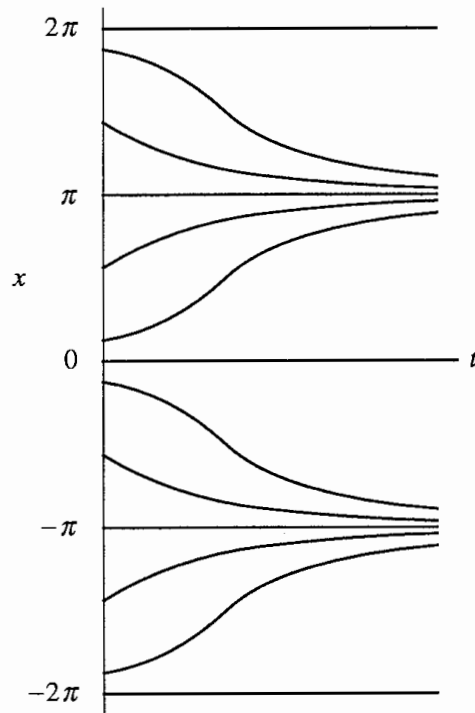


Figure 2.1.3

In all honesty, we should admit that a picture can't tell us certain *quantitative* things: for instance, we don't know the time at which the speed $|\dot{x}|$ is greatest. But in many cases *qualitative* information is what we care about, and then pictures are fine.

2.2 Fixed Points and Stability

The ideas developed in the last section can be extended to any one-dimensional system $\dot{x} = f(x)$. We just need to draw the graph of $f(x)$ and then use it to sketch the vector field on the real line (the x -axis in Figure 2.2.1).

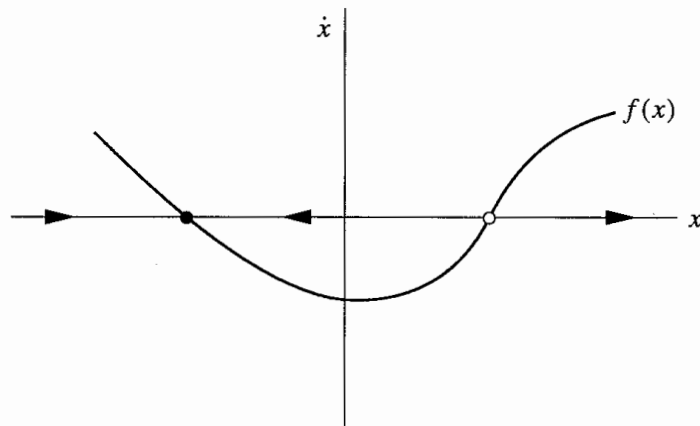


Figure 2.2.1

As before, we imagine that a fluid is flowing along the real line with a local velocity $f(x)$. This imaginary fluid is called the phase fluid, and the real line is the phase space. The flow is to the right where $f(x) > 0$ and to the left where $f(x) < 0$. To find the solution to $\dot{x} = f(x)$ starting from an arbitrary initial condition x_0 , we place an imaginary particle (known as a *phase point*) at x_0 and watch how it is carried along by the flow. As time goes on, the phase point moves along the x -axis according to some function $x(t)$. This function is called the *trajectory* based at x_0 , and it represents the solution of the differential equation starting from the initial condition x_0 . A picture like Figure 2.2.1, which shows all the qualitatively different trajectories of the system, is called a *phase portrait*.

The appearance of the phase portrait is controlled by the fixed points x^* , defined by $f(x^*) = 0$; they correspond to stagnation points of the flow. In Figure 2.2.1, the solid black dot is a stable fixed point (the local flow is toward it) and the open dot is an unstable fixed point (the flow is away from it).

In terms of the original differential equation, fixed points represent *equilibrium* solutions (sometimes called steady, constant, or rest solutions, since if $x = x^*$ initially, then $x(t) = x^*$ for all time). An equilibrium is defined to be stable if all sufficiently small disturbances away from it damp out in time. Thus stable equilibria are represented geometrically by stable fixed points. Conversely, unstable equilibria, in which disturbances grow in time, are represented by unstable fixed points.

EXAMPLE 2.2.1:

Find all fixed points for $\dot{x} = x^2 - 1$, and classify their stability.

Solution: Here $f(x) = x^2 - 1$. To find the fixed points, we set $f(x^*) = 0$ and solve for x^* . Thus $x^* = \pm 1$. To determine stability, we plot $x^2 - 1$ and then sketch the vector field (Figure 2.2.2). The flow is to the right where $x^2 - 1 > 0$ and to the left where $x^2 - 1 < 0$. Thus $x^* = -1$ is stable, and $x^* = 1$ is unstable. ■

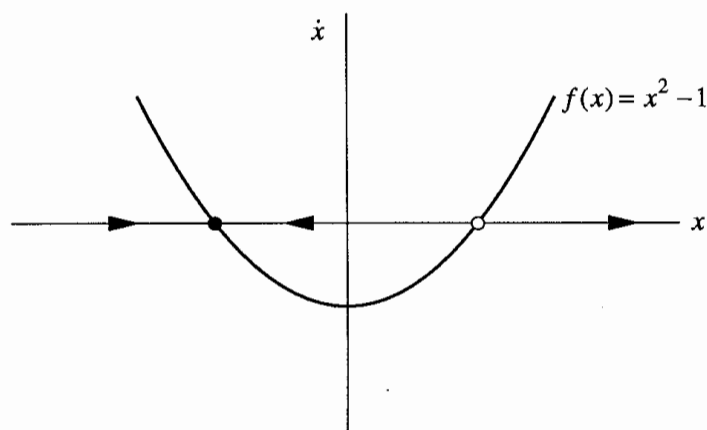


Figure 2.2.2