Review : So far

- we introduced dyn. sys. $\dot{x} = f(x)$
- And learned phase portrait method
- Today: linearization

explains local behavior of a nonlinear system by a linear system.

Plan for today:

- review lin. sys.
- Taylor expansion
- Linearization procedure
- Stability result.

Review of lin. sys.:

1-dim example:

$$\dot{x} = a x$$

money in bank

interest rate / inflation rate

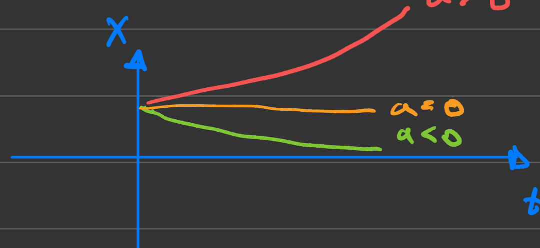
$+$ $\quad$ $-$

- The solution is explicitly known:

$$x(t) = e^{ta} x(0) \quad \text{because}$$

$$\frac{d}{dt} x(t) = \left(\frac{d}{dt} e^{ta}\right) x(0) = a e^{ta} x(0) = a x(t)$$

- The sign of a determines asymptotic behavior as $t \to \infty$.



- if $a > 0$: $\quad x(t) \to \infty$
- if $a = 0$: $\quad x(t) = x(0)$
- if $a < 0$: $\quad x(t) \to 0$

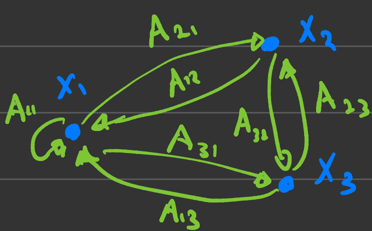
- This observation almost holds in multi demensional case as well.

<u>Multi-dim linear sys.</u> :

- Suppose you want to model the dyn. over a network of n agents (sensor network robots, power grid)

- Let $x_i(t)$ denote the state of i'th agent at time t.
- Its dynamics is linearly related to the state of other agents.

$$\dot{x}(i) = \sum_{j=1}^{n} A_{ij} x_j(t)$$



- We can represent it in matrix-vector notation:

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

- The solution is explicitly known in terms of matrix exponential:

$$X(t) = e^{tA} X(0)$$

$$\frac{d}{dt} e^{tA} = A e^{tA}$$

why?

$$\frac{d}{dt} X(t) = \frac{d}{dt} e^{tA} X(0) = A e^{tA} X(0) = A X(t) \checkmark$$

- The asymptotic behavior of $X(t)$ depends on the eigen values of A.

- We say A is Hurwitz if all eig. values have negative real part:

$$\mathrm{Re}(\lambda) < 0, \quad \forall \text{ eig. value } \lambda$$

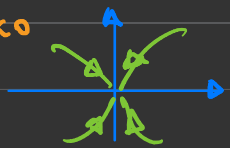
- Main result in lin. sys. theory:

$$\lim_{t \to \infty} X(t) = 0, \ \forall X(0) \iff A \text{ is Hurwitz}$$

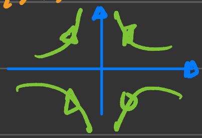
- Example:

① $A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \Rightarrow x(t) = c_1 e^{a_1 t} + c_2 e^{a_2 t}$

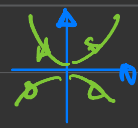
$a_1, a_2 < 0$



$a_2 > 0, a_1 < 0$



$a_1, a_2 > 0$

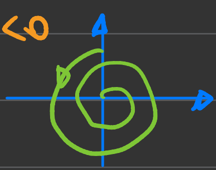


③ $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \Rightarrow x(t) = c e^{(a-ib)t} + c^* e^{(a+ib)t}$

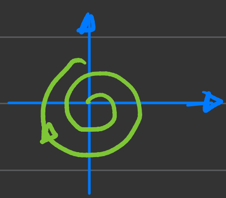
rotate (b)   Scale (a)

$= c_1 e^{at} \sin(bt) + c_2 e^{at} \cos(bt)$

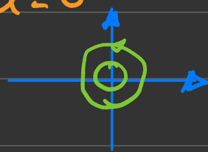
$a < 0$



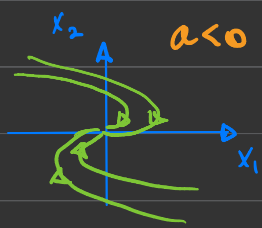
$a > 0$



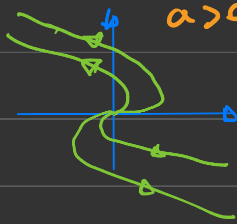
$a = 0$



③ $A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \Rightarrow x(t) = (c_1 + c_2 t) e^{at}$

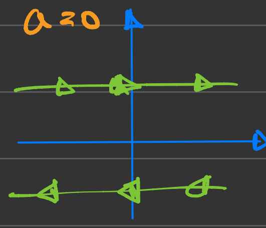


$a < 0$



$a > 0$

$a = 0$

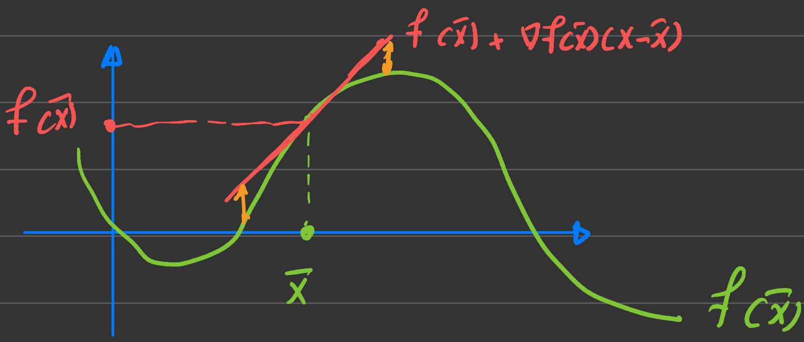


First-order Taylor series expansion: (1-dim)

- $f: \mathbb{R} \to \mathbb{R}$, and $f \in C^2$ $\rightsquigarrow$ twice diff. with cont. derivative

Then,

$$f(x-\bar{x}) = \underline{f(\bar{x}) + f'(\bar{x})(x-\bar{x})} + o(|x-\bar{x}|^2)$$



- Extension to multi-dim. $f: \mathbb{R}^n \to \mathbb{R}^m$

First-order Taylor series expansion: (multi-dim)

- $f: \mathbb{R}^n \to \mathbb{R}^m$, and $f \in C^2$. Then,

$$f(x-\bar{x}) = f(\bar{x}) + \underbrace{\frac{\partial f}{\partial x}(\bar{x})(x-\bar{x})}_{\text{linear part}} + o(\overset{\text{norm}}{\|x-\bar{x}\|^2})$$

where

$$\underset{\text{Jacobian}}{\frac{\partial f}{\partial x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}, & \frac{\partial f_1}{\partial x_2}, & \ldots, & \frac{\partial f_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_m}{\partial x_1}, & \ldots & , & \frac{\partial f_m}{\partial x_n} \end{bmatrix}_{m \times n}$$

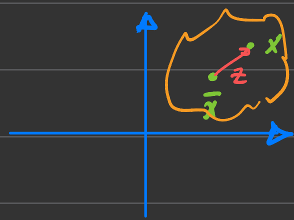
- With this results, we are ready to formalize the linearization procedure:

- linearization of a general nonlinear dyn.

system:

$$\dot{x} = f(x), \qquad f: \mathbb{R}^n \to \mathbb{R}^n$$

- Say $\bar{x}$ is an eqlb. point



therefore $f(\bar{x}) = 0$

- define $z(t) = x(t) - \bar{x}$ and assume z is small.

$$\dot{z} = f(\bar{x} + z)$$

ignore because it is small

Taylor expansion

$$= \underbrace{f(\bar{x})}_{0 \checkmark} + \underbrace{\nabla f(\bar{x})}_{A} z + O(\|z\|^2)$$

$$\simeq A z$$

$\Rightarrow$

$$\boxed{\begin{array}{l} \dot{x} = f(x) \\ f(\bar{x}) = 0 \end{array} \xrightarrow[\text{around } \bar{x}]{\text{linearize}} \begin{array}{l} \dot{z} = A z \text{ where} \\ A = \nabla f(\bar{x}) \end{array}}$$

Example : (pendulum)

$$\dot{x} = \begin{bmatrix} x_2 \\ -\omega^2 \sin(x_1) - \gamma x_2 \end{bmatrix} =: f(x)$$

- eqlb. points

$$f(x) = 0 \Rightarrow \bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or } \hat{x} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

- linearize around $\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\nabla f(x) = \begin{bmatrix} 0 & 1 \\ -\omega^2 \cos(x_1) & -\gamma \end{bmatrix} \Rightarrow \nabla f(\bar{x}) = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{bmatrix}$$

linearization

$$\Rightarrow \quad \dot{z} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{bmatrix} z$$

- Around $\hat{x} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$: $\nabla f(\hat{x}) = \begin{bmatrix} 0 & 1 \\ \omega^2 & -\gamma \end{bmatrix}$

$$\Rightarrow \quad \dot{z} = \begin{bmatrix} 0 & 1 \\ \omega^2 & -\gamma \end{bmatrix} z$$

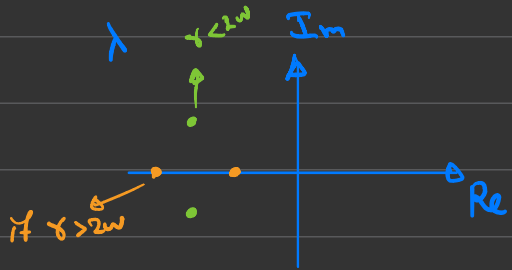
- So what? after linearization, we use our knowledge from linear sys. theory to conclude about local behavior of nonlinear sys.

Pendelum example (continued):

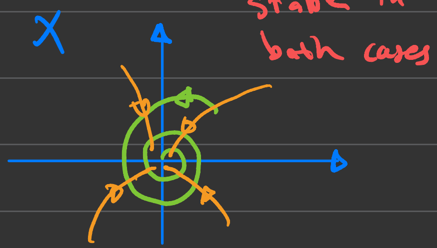
- eqlb. point 1: $\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \dot{z} = Az \quad \text{with} \quad A = \begin{bmatrix} 0 & 1 \\ -w^2 & -\gamma \end{bmatrix}$$

charecterist eq.: $\lambda^2 + \gamma\lambda + w^2 = 0$



$\longrightarrow$

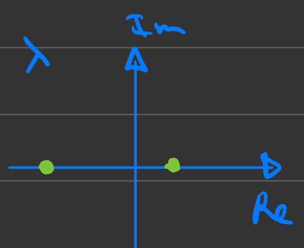


stable in both cases

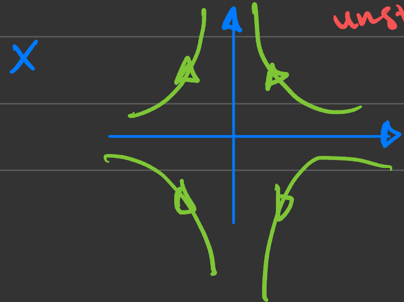
- eqlb. point 2: $\bar{x} = \begin{bmatrix} 0 \\ \pi \end{bmatrix}$

$$\Rightarrow \dot{z} = Az \quad \text{with} \quad A = \begin{bmatrix} 0 & 1 \\ w^2 & -\gamma \end{bmatrix}$$

char. eq.: $\lambda^2 + \gamma\lambda - w^2 = 0$



$\longrightarrow$



unstable

Thm :

- let $\bar{x}$ be eqlb. point of $\dot{x} = f(x)$ and

$A = \nabla f(\bar{x})$.

① if $Re(\lambda) < 0$ for all eig. values of A

then $\bar{x}$ is A.S.

② if $Re(\lambda) > 0$ for any eig. value of A

then $\bar{x}$ is not "stable"

③ if $Re(\lambda) \leq 0$ for all but $Re(\lambda) = 0$ for some eig. values of A. then no conclusion can be made!