First-order Taylor expansion: Assume $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable with Jacobian $\frac{\partial f}{\partial x}$. For any point $\bar{x} \in \mathbb{R}^n$, consider the first-order Taylor approximation

$$f(x) \approx f(\bar{x}) + \frac{\partial f}{\partial x}(\bar{x})(x - \bar{x}).$$

Define the approximation error as

$$R(x, \bar{x}) := f(x) - f(\bar{x}) - \frac{\partial f}{\partial x}(\bar{x})(x - \bar{x}).$$

- If the derivative is continuous, i.e. $f \in \mathcal{C}^1$, then

  $$\lim_{x \to \bar{x}} \frac{\|R(x, \bar{x})\|}{\|x - \bar{x}\|} = 0 \quad \text{or} \quad R(x, \bar{x}) = o(\|x - \bar{x}\|) \tag{0.1}$$

  The little $o(\|x - \bar{x}\|)$ notation means that the error converges to zero faster than a linear rate $\|x - \bar{x}\|$.

- If f is twice-differentiable with continuous second-derivative, i.e. $f \in \mathcal{C}^2$, then there exists a constant C and a radius r such that for all $x \in B_r(\bar{x}) := \{x \in \mathbb{R}^n; \|x - \bar{x}\| \leq r\}$

  $$\|R(x, \bar{x})\| \leq C\|x - \bar{x}\|^2, \quad \text{or} \quad R(x, \bar{x}) = O(\|x - \bar{x}\|^2) \tag{0.2}$$

  The big $O(\|x - \bar{x}\|^2)$ notation means that the error converges to zero with a quadratic rate $\|x - \bar{x}\|^2$.

- If the second-order derivative is uniformly bounded for all $x \in \mathbb{R}^n$, then the error bound (0.2) holds for all $x \in \mathbb{R}^n$ (i.e. the radius r can be selected to be ∞).

Example 1. *Consider the example $f(x) = x\sqrt{x}$. Its first-order and second-order derivatives are $f'(x) = \frac{3}{2}\sqrt{x}$ and $f''(x) = \frac{3}{4\sqrt{x}}$. Therefore $f \in \mathcal{C}^1$ but $f \notin \mathcal{C}^2$. Its Taylor approximation around $\bar{x} = 0$ is*

$$f(x) \approx f(0) + f'(0)x = 0.$$

The approximation error

$$|R(x, 0)| = |x\sqrt{x}|$$

satisfies the limit condition (0.1)

$$\lim_{x \to 0} \frac{R(x, 0)}{|x|} = \lim_{x \to 0} \sqrt{x} = 0$$

while it does not satisfy the bound (0.2)

$$|R(x, 0)| = |x\sqrt{x}| \nleq Cx^2$$

for any constant C.