Fundamentals of ODE:

- Dyn systems are modeled with diff. eq.

$$\dot{x} = f(x) \longrightarrow \text{assume n-dim state } x(t) \in \mathbb{R}^n$$

- We understand linear diff. eq. very well.

$$\dot{x} = Ax \longrightarrow \underbrace{x(t) = e^{tA} x(0)}$$

a unique solution exists for all t.

- However, nonlinear diff eq. have some subtleties.

  $\downarrow$

  Sometimes, solution does not exist or multiple solutions exist.

- Goal: establish sufficient condition for $f(x)$ s.t. a unique solution always exist.

- plan: ① examples ② Lipschitz functions ③ Lip. Lemma ④ Existence & uniqueness

Example 1:

- Imagine an integrator $\dot{x} = u$ with control law $u = \sqrt{x}$ starting from $x(0) = 0$.

$$\dot{x} = \sqrt{x}, \quad x(0) = 0$$

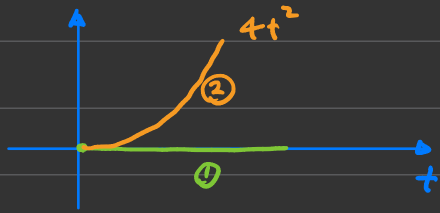
- What is the solution?

① $x(t) = 0 \rightarrow$ No motion

$$\dot{x}(t) = 0 = \sqrt{x(t)}, \quad x(0) = 0$$

② $x(t) = \frac{1}{4}t^2 \rightarrow$ goes to ∞

$$\dot{x}(t) = \frac{t}{2} = \sqrt{x(t)}, \quad x(0) = 0$$



- We have two valid solutions with two diff behavior

- Which one?

Example 2:

- Now consider a quadratic control law:

$$\dot{x} = r x^2, \qquad x(0) = 1$$

- Then, what is the solution at time t?

- We use method of separation to solve this ODE

$$\frac{dx}{dt} = r x^2 \Rightarrow \frac{dx}{x^2} = r\, dt$$

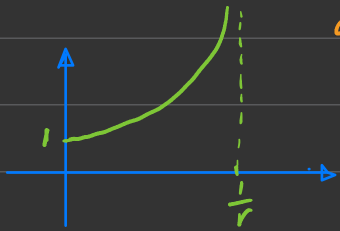
$$\Rightarrow \frac{-1}{x(t)} + \frac{1}{x(0)} = r t$$

$$\Rightarrow x(t) = \frac{1}{1 - rt}$$

$\rightarrow \infty$ as $t \rightarrow \frac{1}{r}$

Solution blows up as $t \rightarrow \frac{1}{r}$

- What is the solution when $t > \frac{1}{r}$?



Example 3:

- Consider the integrator $\dot{x} = u$

and you can only choose control $u = +1$ or $u = -1$

in order to stabilize the sys.

- Consider the case $\underbrace{u = -\operatorname{sgn}(x)}_{\text{sign function}} = \begin{cases} -1 & \text{if } x > 0 \\ +1 & \text{if } x \leq 0 \end{cases}$

- No solution exists when $x(0) = 0$.

- We prove this by showing a contradiction. Assume solution exists. Then,

$$\dot{x}(0) = -\operatorname{sgn}(\underbrace{x(0)}_{0}) = 1 \Rightarrow x(t) > 0 \text{ for small } t$$

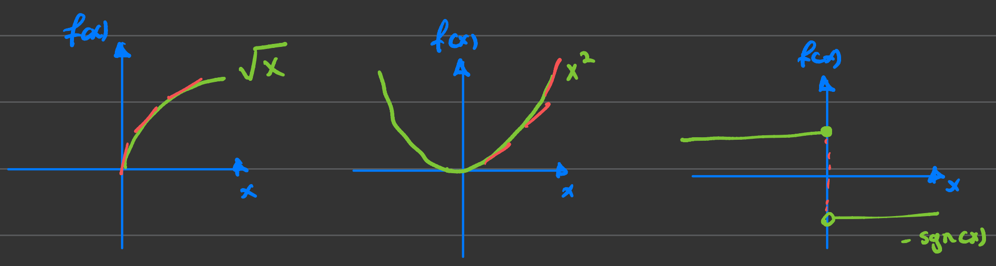
however $x(t) = \int_0^t \dot{x}(s)\,ds = -\int_0^t \underbrace{\operatorname{sgn}(x(s))}_{1}\,ds = -t < 0$

Contradiction

No solution exists.

- These examples show that we need to be careful when we are writing diff. equations.

- We will restrict the class of functions $f(x)$ in order to prevent these cases from happening



① slope becomes ∞ as $x \to 0$

② slope becomes ∞ as $x \to \infty$

③ slope is ∞ at $x = 0$

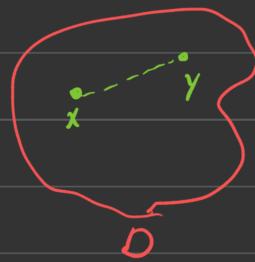
- All three examples happen when slope of $f(x)$ goes to ∞.

- In order to prevent this, we restrict $f(x)$ to a class of <u>Lipschitz</u> functions

Def: $f: \mathbb{R}^n \to \mathbb{R}^m$ is Lip. on $D \subseteq \mathbb{R}^n$ if $\exists\, L > 0$ s.t.

$$\|f(x) - f(y)\| \leq L\|x-y\|, \quad \forall x,y \in D$$

this can be any norm in $\mathbb{R}^n$



- f is locally Lip. if it is Lip. for all bounded and closed sets D.

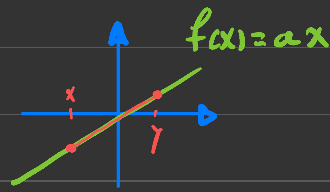
- f is globally Lip. if it is Lip. on $\mathbb{R}^n$.

Remark:

- choice of norm does not affect Lip. property but the value of L.

Examples:

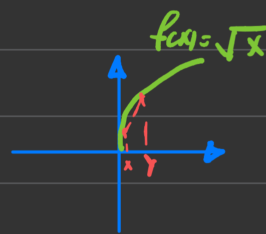
① $f(x) = ax$. To check Lip.



$$\frac{|f(y)-f(x)|}{|y-x|} = \frac{|ay-ax|}{|y-x|} \leq \frac{|a|\,|y-x|}{|y-x|} \leq \underbrace{|a|}_{L}$$

$\Rightarrow |f(y)-f(x)| \leq |a|\,|y-x|, \ \forall x,y \in \mathbb{R}$ $\rightarrow$ globally Lip.

② $f(x) = \sqrt{x}$, $x \geq 0$

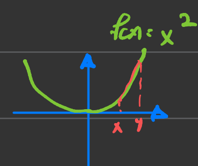


$$\frac{|f(y)-f(x)|}{|y-x|} = \frac{\cancel{|\sqrt{y}-\sqrt{x}|}}{\cancel{|\sqrt{y}-\sqrt{x}|}\,|\sqrt{y}+\sqrt{x}|} = \frac{1}{\sqrt{x}+\sqrt{y}} \rightarrow \infty \text{ as } x,y \rightarrow 0$$

$\Rightarrow$ Not Lip. on $[0,\infty)$ but Lip. on $[a,\infty)$ for $a>0$

$$\frac{1}{\sqrt{x}+\sqrt{y}} > \underbrace{\frac{1}{2\sqrt{a}}}_{L} \text{ for all } x,y \geq a$$

③ $f(x) = x^2$



$$\frac{|f(y)-f(x)|}{|y-x|} = \frac{|y^2-x^2|}{|y-x|} = |y+x| \rightarrow \infty \text{ as } x,y \rightarrow \infty$$

$\Rightarrow$ Not globally Lip., but locally Lip. because when $|x|, |y| \leq M$

then $|x+y| \leq 2M \rightarrow L$

<u>Observation:</u> Lip. depends on derivative.

bdd derivative $\Rightarrow$ Lip.

<u>Lemma:</u> (lemma 3.1 in Khalil)

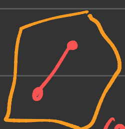
- Assume $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable
- And $\left\| \frac{\partial f}{\partial x}(x) \right\| \le L \quad \forall x \in W$

$\Rightarrow \| f(x) - f(y) \| \le L \| x - y \| \quad \forall x, y \in W$

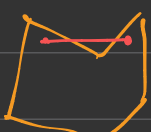
or f is Lip. on W

W: Convex set

Convex set: every two points are connected with a line inside the set



Convex

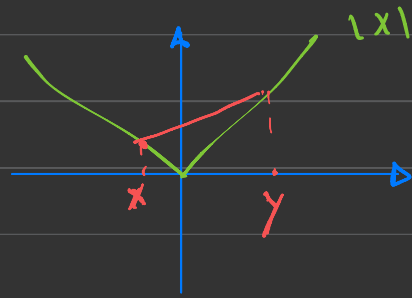


Not convex

- What if f is not diff.?

Example:

- $f(x) = |x|$



Not diff at 0, but still globally Lip.

- How strong is Lip. assumption?