Goal: introduce Lyapunov method for stability analysis

- Stability is an important concept in design of engineering systems.

- The stability of linear systems is well understood

$\dot{x} = Ax$ is stable $\Leftrightarrow$ A is Hurwitz

- For stability analysis of nonlinear systems, we need a new tool $\rightarrow$ Lyapunov method

↳ general and powerful method to ensure stability.

- We start by introducing different notions of stability that might occure in a non linear sys.

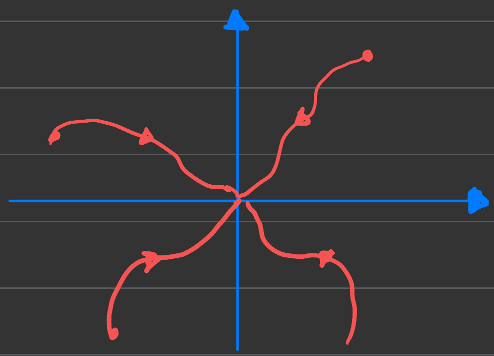
Definitions of Stability :

- Consider the dyn. sys. $\dot{x} = f(x)$ with eqlb. point $\bar{x}$, i.e. $f(\bar{x}) = 0$. $\rightarrow$ WLOG assume $\bar{x} = 0$

③ Globally Asymptotically Stable (GAS)

$$\lim_{t \to \infty} \|x(t)\| = 0, \qquad \forall\, x(0) \in \mathbb{R}^n$$

$\rightarrow$ Starting from any initial condition, all trajectories converge to the eqlb. point.

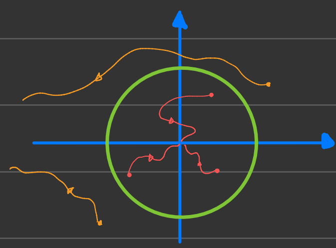


② Asymptotically stable (AS):

Stable and

$\exists \delta > 0$ s.t. $\lim_{t \to \infty} \|x(t)\| = 0$, if $\|x(0)\| \leq \delta$

↳ all trajectories converge to eqlb. if they start close enough.



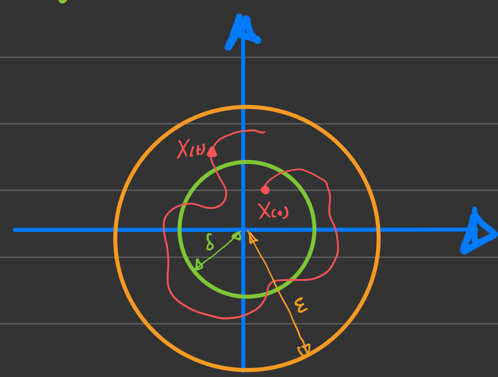
① Stable (S):

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\|x(t)\| \leq \varepsilon$, if $\|x(0)\| \leq \delta$

↳ all trajectories remain arbitrary close to eqlb. if they start close enough

Remark:

GAS $\Rightarrow$ AS $\Rightarrow$ S



Example:

① $\dot{x} = -x$

eqlb. point $\bar{x} = 0 \longrightarrow$ GAS

② lin. sys. : $\dot{x} = Ax$

A is Hurwitz $\Leftrightarrow$ GAS

③ pendelum:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\gamma x_2 - w^2 \sin(x_1) \end{bmatrix}$$

- Eqlb. points: $\bar{x}^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or $\bar{x}^{(2)} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$

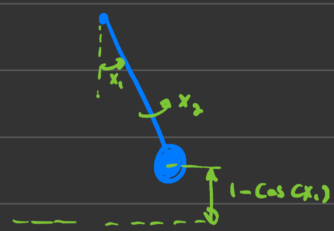
- by looking at phase-portrait:
  - if $\gamma > 0 \Rightarrow \bar{x}^{(1)}$ is AS and $\bar{x}^{(2)}$ is not stable
  - if $\gamma = 0 \Rightarrow \bar{x}^{(1)}$ is S and $\bar{x}^{(2)}$ is not stable

- In the pendelum example, we used phase-portrait to conclude about stability.

- Alternatively, we can use energy concepts to reach the same result.

- Define the energy function:

$$V(x) = \underbrace{\frac{1}{2}x_2^2}_{\text{Kinetict energy}} + \underbrace{\omega^2(1-\cos(x_1))}_{\text{gravitational Potential energy}}$$



- positive and takes its minimum value at eqlb. point $\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Main idea: in order to study convergence to eqlb.,

we study how does energy change with time

$$\frac{d}{dt} V(X(t)) = \frac{d}{dt}\left[\frac{1}{2} X_2^2(t) + w^2(1-\cos(X_1(t)))\right]$$

$$= X_2(t)\,\dot{X}_2(t) + w^2 \sin(X_1(t))\,\dot{X}_1(t)$$

$$= -\gamma X_2(t)^2 \cancel{- w^2 X_2(t) \sin(X_1(t))}$$
$$\cancel{+ w^2 \sin(X_1(t)) X_2(t)}$$

$$= -\gamma X_2(t)^2$$

- if $\gamma > 0$, then $\frac{d}{dt} V(X(t)) < 0$ unless $X(t) = 0$

$\Rightarrow V(X(t))$ decreases unless $X(t) = 0$

$\Rightarrow V(X(t)) \to 0$

$\Rightarrow X(t) \to 0$

AS

will present a rigorous argument later.

- if $\gamma = 0$, then $\frac{d}{dt} V(X(t)) = 0$

$\Rightarrow$ $V(X(t))$ remains constant with time

$\Rightarrow$ $X(t)$ moves along contours of $V(X)$ and contours can be arbitrary small depending on initial condition

$\Rightarrow$ Stable

- Functions that behave like this energy function are called Lyapunov functions.

Lyapunov functions:

– Let $V: \mathbb{R}^n \to \mathbb{R}$ be C^1 ← Continuously differentiable

① We say V is positive definite (p.d.) if

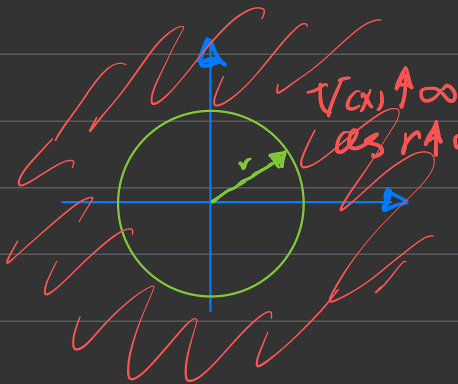
$$V(x) > 0 \quad \forall\, x \neq 0 \quad \text{and} \quad V(0) = 0$$

② We say V is radially unbounded if

$$V(x) \to \infty \quad \text{as} \quad \|x\| \to \infty$$

more precisely

$$\forall M > 0, \exists\, r > 0 \text{ s.t. } V(x) \geq M \text{ if } \|x\| \geq r$$



Example:

– $V(x) = x_1^2 + x_2^2$ → p.d. radially unbdd

– $V(x) = x_1^2 + x_2^2 + 2x_1x_2$
$= (x_1 + x_2)^2$

not p.d.
not radially unbdd

- Consider the dynamics $\dot{x} = f(x)$.

③ rate of change of $V(x(t))$ along the trajectory is

$$\frac{d}{dt} V(x(t)) = \underbrace{\nabla V(x(t))^T f(x(t))}_{\dot{V}(x(t))}$$

we define the function $\dot{V}(x) := \nabla V(x)^T f(x)$

$\dot{V} : \mathbb{R}^n \to \mathbb{R}$

sometimes it is denoted by $d_f V(x)$ to explicitly emphasize the dependence on $f(x)$

Example: $\dot{x} = \underbrace{-x}_{f(x)}$ and $V(x) = x^2$

$$\Rightarrow \quad \dot{V}(x) = \nabla V(x)^T f(x) = 2x(-x) = -2x^2$$

Thm:

- Assume $\bar{x}=0$ is an eqlb. point for $\dot{x}=f(x)$

- And there exists a p.d. C^1 function $V: \mathbb{R}^n \to \mathbb{R}$

- Define $\dot{V}(x) = \nabla V(x)^T f(x) \rightsquigarrow \frac{d}{dt} V(x(t)) = \dot{V}(x(t))$

③ if $\dot{V}(x) < 0 \quad \forall x \in \mathbb{R}^n/\{0\}$ and $\underbrace{V(x) \to \infty \text{ as } \|x\| \to \infty}_{\text{radially unbounded}}$

then $\bar{x}=0$ is GAS

② if $\dot{V}(x) < 0 \quad \forall x \in D/\{0\}$ → open set containing 0

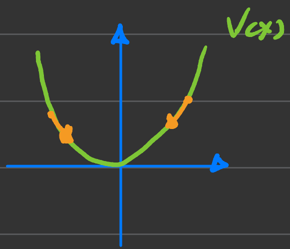
then $\bar{x}=0$ is AS

① if $\dot{V}(x) \leq 0 \quad \forall x \in D$

then $\bar{x}=0$ is S

Remark: the result states that if you find a Lyapunov func then you can conclude about the stability, but does not tell how to find Lyapunov function.

Example:



① $\dot{x} = -x$, $\quad V(x) = x^2$

$V(x) > 0 \ \forall x \neq 0$ and $V(0) = 0 \Rightarrow$ p.d. ✓

$V(x) \to \infty$ as $|x| \to \infty \Rightarrow$ radially unbdd

$\dot{V}(x) = 2x(-x) = -2x^2 < 0 \ \forall x \neq 0 \Rightarrow$ GAS.

② pendulum:

$V(x) = \frac{1}{2}x_2^2 + \omega^2(1-\cos(x_1)) \to$ p.d. ✓

$\dot{V}(x) = -\gamma x_2^2 \leq 0 \Rightarrow$ stable

need more to conclude about AS.